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Parabolic $U(p,q)$-Higgs bundles

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Thesis submitted in partial fulfillment of the requirements
for the degree of European Doctor in Mathematics.
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Madrid, March 2006

A MIS PADRES

Abstract

In this thesis we study parabolic $U(p,q)$-Higgs bundles on a compact Riemann surface with a finite set of marked points. These objects correspond to representations of the fundamental group of the Riemann surface with punctures at the parabolic points in $U(p,q)$, with fixed compact holonomy classes around the marked points.

Our approach combines techniques used by Bradlow, García-Prada and Gothen [BGG] in the non-parabolic case as well as those used by García-Prada, Gothen and Muñoz in [GGM] to study the topology of parabolic $GL(3,\mathbb{C})$-Higgs bundles.

The strategy is to use the Bott–Morse theoretic techniques introduced by Hitchin in [H]. The connectedness properties of the moduli space reduces to the connectedness of a certain moduli space of parabolic triples introduced by Biquard and García-Prada in [BG] in connection with the study of the parabolic vortex equations and instantons of infinite energy. Much of the work is devoted to a thorough study of these moduli spaces of triples and its connectedness properties.

Our main results include the counting of number of connected components of the moduli space of parabolic $U(n,1)$-Higgs bundles as well as the computation of the Poincaré polynomial of the moduli space of parabolic $U(2,1)$-Higgs bundles with one marked point.

Agradecimientos

Quiero agradecer en primer lugar a los directores de esta tesis, Oscar García Prada y Vicente Muñoz Velázquez, su ayuda generosa, su paciencia y su exigencia durante todo este tiempo, sin ellos esta tesis no habría visto nunca la luz.

Agradezco también a Antonio Córdoba su pequeño empujón hacia el mundo de la investigación en Matemáticas. Sin su confianza no habría comenzado este camino. Gracias también a Nigel Hitchin y a Peter Gothen por su hospitalidad y ayuda, tanto en los comienzos de esta tesis como a lo largo de su elaboración. A Luis Álvarez, Ignasi Mundet y a Tomás Gómez por tantas conversaciónes matemáticas y tantas respuestas a mis múltiples preguntas.

También quiero agradecer a mis compañeros de despacho, Alberto, Estela, Marta, Rafa, y de despachos cercanos, Álvaro, Sergio y Susana, tantos cafés compartidos.

He de agradecer también al CSIC su hospitalidad en los últimos años de esta tesis.

Finalmente quiero agradecer a mi familia y a mis amigos, Beatriz, Lourdes, Ana, Marifé, Rocio, Connie, Fabio, Cristina, Mariangel y Amparo, su constante apoyo durante estos años.

Contents

Introduction

Let X be a compact Riemann surface of genus $g \geq 0$ and let $\{x_1, \ldots, x_s\}$ be a finite set of marked points in X. Let $D = x_1 + \cdots + x_s$ be the corresponding effective divisor. A *parabolic bundle* is a holomorphic vector bundle together with a weighted flag,

$$\begin{aligned} E_x &= E_{x,1} \supset \cdots \supset E_{x,r(x)} \\ 0 &\leq \alpha_1(x) < \cdots < \alpha_{r(x)}(x) < 1, \quad x \in D, \end{aligned}$$

at each marked point —the parabolic structure.

Parabolic bundles were introduced by Seshadri [Se] in order to obtain a desingularization of the moduli space of semistable vector bundles of rank two and degree zero [Se2]. The *parabolic degree* and *parabolic slope* of E are defined by

$$\begin{aligned} \operatorname{pardeg}(E) &= \deg(E) + \sum_{x \in D} \sum_{i=1}^{r(x)} k_{x,i}\alpha_i(x), \\ \operatorname{par}\mu(E) &= \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)}, \end{aligned}$$

where $\deg(E)$ is the degree of E, $\operatorname{rk}(E)$ is the rank of E and $k_{x,i} = \dim E_{x,i}/E_{x,i+1}$ is the *multiplicity* of the weight α_i at the marked point x. The parabolic bundle is said to have full flags if $k_{x,i} = 1$ for all $\alpha_i(x)$ and all $x \in D$. A parabolic bundle E is said to be *(semi)stable* if $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (resp. $\operatorname{par}\mu(E') \leq \operatorname{par}\mu(E)$) for every non-trivial parabolic subbundles E' of E. The parabolic bundle E is *polystable* if it is a direct sum of stable parabolic bundles of the same parabolic slope. With this notion Mehta and Seshadri constructed the moduli space of polystable parabolic bundles with fixed parabolic structure and degrees using Mumford's geometric invariant theory [MF]. This moduli space is a normal projective variety which is also smooth for a generic choice of weights, those for which strict semistability does not occur. For sufficiently close generic weights the moduli spaces are isomorphic.

Mehta and Seshadri in [MS] showed that, the moduli space of polystable parabolic bundles with parabolic degree zero and fixed parabolic structure over a Riemann surface of genus $g \geq 2$, can be identified with the moduli space of unitary representations of the fundamental group of $X - D$ with fixed holonomy around the marked points determined by the parabolic structure. A proof of this theorem, using gauge theory, has been given by Biquard [B] and others [P, Bo, NaSt]. Mehta and Seshadri's theorem generalises a theorem of Narasimhan and Seshadri [NS] which identifies, the moduli space of polystable vector bundles of degree zero over a compact Riemann surface, with the space of unitary representations of the fundamental group of the Riemann surface.

Let K be the canonical bundle of X. A *parabolic Higgs bundle* is a pair (E, Φ) where E is a parabolic bundle, and $\Phi : E \to E \otimes K(D)$ is a strongly parabolic homomorphism. This means that Φ is a meromorphic endomorphism valued one-form with simple poles along D whose residue at $x \in D$ is nilpotent with respect to the flag. A parabolic Higgs bundle (E, Φ) is *stable* if $\text{par}\mu\,(E') < \text{par}\mu\,(E)$ for every $E' \subset E$ proper Φ-invariant parabolic subbundle, i.e. $\Phi(E') \subset E' \otimes K(D)$. It is said to be *semistable* if we require the weaker inequality for the slope condition above and, *polystable* if it decomposes as direct sum of stable parabolic Higgs bundles of the same parabolic slope.

The moduli space of parabolic Higgs bundles, $\mathcal{M}$, was constructed using geometric invariant theory by Yokogawa in [Y1, Y2], who also showed that, for generic weights, it is a smooth irreducible quasiprojective complex variety. Analogously to the non parabolic case, the moduli space $\mathcal{M}$ contains the cotangent bundle of the moduli space of stable parabolic bundles.

Simpson in [S2] proves that the moduli space of polystable parabolic Higgs bundles of parabolic degree zero can be identified with the moduli space of polystable representations of the fundamental group of the surface with marked points in $\mathrm{GL}(n, \mathbb{C})$, with fixed compact holonomy around the marked points. This extends to non-compact Riemann surfaces the theory of Higgs bundles and representations of the fundamental group developed by Hitchin, Simpson, Donaldson and Corlette [H, S1, D2, C].

This thesis is devoted to the study of parabolic $\mathrm{U}(p,q)$-Higgs bundles. A *parabolic* $\mathrm{U}(p,q)$-*Higgs bundles* is a parabolic Higgs bundle of the form,

$$E = V \oplus W \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \to (V \oplus W) \otimes K(D),$$

where $\beta : W \to V \otimes K(D)$ and $\gamma : V \to W \otimes K(D)$ are strongly parabolic homomorphisms.

A parabolic $\mathrm{U}(p,q)$-Higgs bundle is said to be stable if the slope condition is satisfied for every proper Φ-invariant subbundles $E' \subset E$ such that $E' = V' \oplus W'$ with $V' \subset V$ and $W' \subset W$, i.e for all proper subbundles parabolic subundles $V' \subset V$ and $W' \subset W$ such that,

$$\gamma(V') \subset W' \otimes K(D)$$
$$\beta(W') \subset V' \otimes K(D).$$

Semistability is defined by requiring the weaker inequality instead of the strict inequality, and *polystable* if it is a sum of stable parabolic $\mathrm{U}(p',q')$-Higgs bundles of the same slope.

Let $\mathcal{U}(p,q,a,b;\alpha,\eta)$ be the moduli space of parabolic $\mathrm{U}(p,q)$-Higgs bundles with fixed ranks $\mathrm{rk}\,(V) = p$, $\mathrm{rk}\,(W) = q$, degrees $\deg(V) = a$, $\deg(W) = b$ and systems of weights α and η for V and W. The moduli space $\mathcal{U}(p,q,a,b;\alpha,\eta)$ is a closed subvariety of the moduli space $\mathcal{M}(p+q,a+b;\alpha\cup\eta)$, where $\alpha\cup\eta$ is the system of weights given by the direct sum of the parabolic bundles V and W, and it is isomorphic to the moduli space of irreducible representations of the fundamental group of the surface in $\mathrm{U}(p,q)$, with fixed holonomy classes in the maximal compact subgroup $\mathrm{U}(p)\times\mathrm{U}(q)$ of $\mathrm{U}(p,q)$, around the marked points.

The goal of this thesis is to count the connected components of $\mathcal{U}(n,1,a,b;\alpha,\eta)$ the moduli space of $\mathrm{U}(n,1)$ parabolic Higgs bundles. The non-parabolic case was studied in [BGG] where the number of connected components of the moduli space of $\mathrm{U}(p,q)$-Higgs bundles was computed.

The *parabolic Toledo invariant* τ for a parabolic $\mathrm{U}(p,q)$-Higgs bundle is defined as

$$\tau = 2\frac{q\,\mathrm{pardeg}\,(V) - p\,\mathrm{pardeg}\,(W)}{p+q}$$

using stability, one shows that

$$|\tau| < \min\{p,q\}(2g-2+s),$$

which generalises the Milnor–Wood inequality of the non-parabolic case.

When we restrict to $q=1$ we find a better bound for the Toledo invariant. In fact, we find the maximal value of the parabolic Toledo invariant, τ_L, which is $\tau_L \leq (2g-2+s)$, where the equality holds when we have no marked points on the surface.

The main result of this thesis is the following.

Theorem. *Let X be a compact Riemann surface with a set of marked points of genus $g \geq 1$, and let α and η be generic weights. Then, the moduli space $\mathcal{U}(n,1,a,b;\alpha,\eta)$ of parabolic $\mathrm{U}(n,1)$-Higgs bundles with full flags is non-empty and connected if and only if $|\tau| < \tau_L$.*

Our approach to the study of $\mathcal{U}$ combines the techniques used in [BGG] to study $\mathrm{U}(p,q)$-Higgs bundles in the non-parabolic case as well as those used in [GGM] to study the topology of moduli spaces of $\mathrm{GL}(3,\mathbb{C})$-parabolic Higgs bundles. The main tool for the study of these moduli spaces is the use of Morse-theoretic techniques introduced by Hitchin in [H]: the L^2-norm of the Higgs field is the moment map f associated to a Hamiltonian action of the circle on the moduli space of solutions to the Hitchin's equations. Such map f is proper and bounded below, hence if the subspace of local minima of f is connected so is $\mathcal{U}$.

The local minima of f are related with another kind of parabolic objects: parabolic triples. A *parabolic triple* is a triple $T = (E_1, E_2, \phi)$ consisting of two parabolic bundles E_1 and E_2 and a strongly parabolic morphism $\phi : E_2 \to E_1(D)$. For a parameter $\sigma \in \mathbb{R}$ we define the *parabolic σ-slope* of a parabolic triple as,

$$\mathrm{pardeg}\,(T) = \frac{\mathrm{pardeg}\,(E_1) + \mathrm{pardeg}\,(E_2)}{\mathrm{rk}\,(E_1) + \mathrm{rk}\,(E_2)} + \sigma\frac{\mathrm{rk}\,(E_2)}{\mathrm{rk}\,(E_1) + \mathrm{rk}\,(E_2)}.$$

A parabolic triple is called *σ-stable* if the slope condition $\mathrm{par}\mu_\sigma(T') < \mathrm{par}\mu_\sigma$ is satisfied for all proper subtriples T' of T. Semistability is defined by requiring the weaker inequality instead of strict inequality. The triple is said to be *σ-polystable* if it can be written as direct sum of σ-stable parabolic triples.

We denote $\mathcal{N}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ the moduli space of σ-stable parabolic triples of fixed ranks $\mathrm{rk}\,(E_1) = r_1$, $\mathrm{rk}\,(E_2) = r_2$, degrees $\deg(E_1) = d_1$, $\deg(E_2) = d_2$, and weights α^1 and α^2.

It turns out that the subvariety of local minima of f can be identified with a certain moduli space of parabolic triples for $\sigma = 2g - 2$. Hence, we focus our work on the study of the connected components of this moduli space. This is done by studying how the moduli spaces change as the stability parameter σ changes. Fix the topological type of the triple $(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$. We call σ_c a *critical value* for σ if there is a σ-semistable triple with such topological type. We call *flip loci* to the subset of $\mathcal{N}_{\sigma_c^\pm}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ of $\sigma_c^\pm$-stable triples but $\sigma_c^\mp$-unstable triples, where $\sigma^\pm = \sigma_c \pm \epsilon$ for $\epsilon > 0$ in $\mathbb{R}$ such that there is no other critical value in $[\sigma_c, \sigma_c^\pm]$. It is the space that has to added to the moduli space of $\mathcal{N}_{\sigma_c}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ when we cross a critical value of σ. We prove that for $\sigma \geq 2g - 2$ the codimension of the flip loci is positive, hence, the moduli spaces $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ are birational for all values of $\sigma \geq 2g - 2$.

Theorem. *Let X be a Riemann surface with a finite set of marked points and genus $g \geq 1$. Let $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ be the moduli space of σ-stable parabolic triples with full flags on* *its parabolic structures. Then, for $\sigma \in [2g-2, \sigma_L)$ the moduli spaces $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ are birational.*

Similarly to the parabolic toledo invariant, using stability one can shows that in the case $r_1 \neq r_2$ there are bounds for sigma $\sigma_m < \sigma < \sigma_M$. such that out of this bounds the moduli space is empty. The upper bound can be improved. We give the maximal value for σ, denote it σ_L, in Proposition 2.7.9. In the case $r_1 = r_2$, the parameter σ is not bounded above, in such case σ_L is the largest critical value of σ.

This maximal value is such that for σ_L^+ the moduli space $\mathcal{N}_{\sigma_L^+}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ is empty and non-empty for σ_L^-. The birationality of these moduli spaces implies that it suffices to count the connected components for the moduli space $\mathcal{N}_{\sigma_L^-}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$. We give an explicit description of this moduli space for $r_1 = n$ and $r_2 = 1$, which leads to the following theorem.

Theorem. *Let X be a compact Riemann surface with genus $g \geq 1$. Let D be a effective divisor associated to a finite set of marked points on X. Then the moduli space $\mathcal{N}_{\sigma_L^-}(n, 1, d_1, d_2; \alpha^1, \alpha^2)$ of σ_L^--stable parabolic triples with generic weights and full flags on its parabolic structures, is an irreducible algebraic variety.*

The moment map f also defines a perfect Bott–Morse function on the moduli space. The computation of the Poincaré series and the indices of all the critical submanifolds of this function gives us the Betti numbers of $\mathcal{U}$. The critical subvarieties of f are : the minima, identified as parabolic triples of ranks $(2,1)$ and $(1,2)$, and parabolic chains with ranks $(1,1,1)$, where a parabolic chain is defined as an n-tuple of parabolic bundles E_i together with a $(n-1)$ tuple of strongly parabolic homomorphisms $\phi_i : E_i \to E_{i-1}(D)$. We thus need to compute the Poincaré polynomial of the moduli space of parabolic chains. In the case of ranks $p = 2$ and $q = 1$, this computation can be done since the moduli space of parabolic chains of ranks $(1,1,1)$ is isomorphic to $\mathrm{Jac}\,(X)^d \times S^{m_1}X \times S^{m_2}X$ for certain d, m_1 and m_2, whose Poincaré polynomial is known [M]. For arbitrary (p,q) the computation of the Poincaré polynomial of the moduli spaces of parabolic chains is much more dificult and it is out of the scope of this thesis. The Poincaré polynomial of the moduli spaces of triples of ranks $(2,1)$ and $(1,2)$ are computed here adapting the results of [GGM].

Hence, we compute the Poincaré polynomial of $\mathcal{U}(2,1,a,b;\alpha,\eta)$ for the case of one marked point, see Theorem 4.6.1. As mentioned above the moduli space $\mathcal{U}(2,1,a,b;\alpha,\eta)$ is a subvariety of the moduli space $\mathcal{M}(3, a+b, \alpha \cup \eta)$. It is known that, the moduli spaces

$\mathcal{M}(3, a+b, \alpha \cup \eta)$ of stable parabolicGL$(3, \mathbb{C})$-Higgs bundles has the same Betti numbers for different choices of degrees and weights (see [GGM, T]). Our computations produce counterexamples to this type of phenomena in the U$(2,1)$-case since, as shown in Theorem 4.6.1, the Betti numbers of the moduli space $\mathcal{U}(2,1,a,b;\alpha,\eta)$ depend on the parabolic structure.

We give now an outline of the thesis. In Chapter 1 we study basic facts on parabolic bundles and parabolic homomorphisms. We introduce the correspondence given by Mehta and Seshadri's theorem.

Chapter 2 is devoted to parabolic triples and parabolic homomorphisms of triples. We recall from [GGM] the theory of extensions and deformations of parabolic triples as well as the notion of the *flip loci* in order to study how the moduli space of stable parabolic triples changes as the parameter of stability σ varies. Next, we give a bound for the codimension of the flip loci. To do this, we exploit the correspondence between stable parabolic triples and parabolic vortex equations. We prove that the codimension of this flip loci is positive whenever $\sigma \in [2g-2, \sigma_L)$ and that it is zero when $\sigma = \sigma_L$. There the moduli space is empty and becomes non empty at σ_L^-. Finally, we describe explicitly the moduli space $\mathcal{N}_{\sigma_L^-}(n, 1, d_1, d_2; \alpha^1, \alpha^2)$.

Chapter 3 is concerned with the study of parabolic U(p,q)-Higgs bundles. We study here the bounds for the parabolic Toledo invariant and find its maximal value when $q = 1$. We also show the Morse theoretic techniques needed for the study of $\mathcal{U}$, and we give the correspondence between the moduli spaces $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ of parabolic triples and the submanifold of minima of the Morse function. This fact lead us to our previous study on the moduli spaces of stable parabolic triples. We show the relationship of $\mathcal{U}(p,q,a,b;\alpha,\eta)$ with the moduli space of representations of the fundamental group of the non-compact Riemann surface $X - D$ on $U(p,q)$ with fixed holonomy in U$(p) \times$ U(q).

Finally in Chapter 4 we fix the ranks $p = 2$ and $q = 1$, and use Bott–Morse theory in order to compute Betti numbers of $\mathcal{U}$ when $p + q = 3$, and one marked point.

Chapter 1

Parabolic bundles

1.1 Definitions and basic facts.

Let X be a compact Riemann surface of genus $g \geq 0$ and let $\{x_i\}_{i=1}^{s} \subset X$ be a finite set of marked points in X. Let $D = x_1 + \ldots + x_s$ be the corresponding effective divisor.

Let E be a holomorphic vector bundle over X.

Definition 1.1.1. A *quasi-parabolic structure on* E is a filtration on each fibre E_x of the bundle E, where $x \in D$, i.e.

$$E_x = E_{x,1} \supset E_{x,2} \supset \cdots \supset E_{x,r(x)+1} = \{0\} \quad \text{for every } x \in D.$$

Definition 1.1.2. A *parabolic structure on* E is given by a quasi-parabolic structure together with a set of real vectors, $\{\alpha(x) = (\alpha_{x,1}, \ldots, \alpha_{x,r(x)})\}_{x\in D}$ such that

$$0 \leq \alpha_{x,1} < \cdots < \alpha_{x,r(x)} < 1 \quad \text{for every } x \in D.$$

This set of vectors is called *weight type* of E.

The *multiplicity of the weight* $\alpha_{x,i}$ is defined as $k_{x,i} = \dim(E_{x,i}/E_{x,i+1})$. When the multiplicity of the weights is one for all weights over a point x we say that the filtration over the point x is a *full flag.* It will sometimes be convenient to repeat each weight according to its multiplicity, i.e., we set $\tilde{\alpha}_1(x) = \cdots = \alpha_{k_{x,i}}(x) = \alpha_1(x)$, etc. We then have weights $0 \leq \tilde{\alpha}_1(x) \leq \cdots \leq \tilde{\alpha}_r(x) < 1$, where $r = \mathrm{rk}\,(E)$ is the rank of E.

A *trivial parabolic structure* on a bundle E consists of a null system of weights and a filtration $E_x = E_{1,x} \supset \{0\}$ over each marked point.

Definition 1.1.3. A *parabolic bundle* E on X is a holomorphic vector bundle with a parabolic structure on it.

Example 1.1.4. Let $X = \mathbb{C}$ and $x = 0$. The trivial line bundle $X \times \mathbb{C}$ with the trivial parabolic structure, i.e. the filtration is $\mathbb{C} \supset \{0\}$ over x and the system of weights is $\alpha(x) = 0$ for all $x \in D$, is a parabolic line bundle over X.

Definition 1.1.5. Let E be a parabolic bundle over the Riemann surface X with marked points on D, and weight type α. The *parabolic dual* E^* is, by definition, the bundle $\operatorname{Hom}(E, \mathcal{O}(-D))$, with the following parabolic structure

$$E^*_x = E^*_{x,1} \supset \cdots \supset E^*_{x,s(x)} \supset \{0\}$$

with $E^*_{x,i} = \operatorname{Hom}(E_x/E_{x,s(x)+2-i}, \mathcal{O}(-D)_x)$ and weights $1 - \alpha_{s(x)}(x) < \cdots < 1 - \alpha_1(x)$.

Definition 1.1.6. Let E and F be parabolic bundles over X, and let α, η be the system of weights at E and F. A *(strongly) parabolic homomorphism* from E to F over X is, a holomorphic homomorphism $f : E \to F$ such that

$$\text{if } \alpha_{x,i} > \eta_{x,j} \text{ (resp. } \alpha_{x,i} \geq \eta_{x,j}) \quad \text{then} \quad f_x(E_{x,\alpha_i}) \subseteq F_{x,\eta_{j+1}}. \tag{1.1}$$

Example 1.1.7. Let X be a Riemann surface with one marked point x on it. Let $E = X \times \mathbb{C}^3$ and $F = X \times \mathbb{C}^4$ be the trivial bundles of ranks 3 and 4, respectively. Consider the following parabolic structures on them. Full flags for E and F over x and weight types α and β for E and F such that

$$\eta_1 < \alpha_1 < \alpha_2 = \eta_2 < \eta_3 < \alpha_3 = \eta_4 < 1.$$

Thence a parabolic homomorphism f from E to F must be such that

$$f(E_1) \subset F_2, \; f(E_2) \subset F_2, \; f(E_3) \subset F_4,$$

and a strongly parabolic homomorphism g from E to F must be such that

$$g(E_1) \subset F_2, \; g(E_2) \subset F_3, \; g(E_3) = 0.$$

Definition 1.1.8. Let E, F be parabolic bundles over X. We define $\operatorname{ParHom}(E, F)$ the *sheaf of parabolic homomorphisms from E to F* to be the sheaf of germs of parabolic homomorphisms from E to F, that is, for each U open subset of X

$$\operatorname{ParHom}(E, F)(U) = \{f \in \operatorname{Hom}(E, F)(U)/f \text{ is parabolic}\}.$$

Analogously, the *sheaf of strongly parabolic morphisms* $\mathrm{SParHom}\,(E,F)$ is such that for each open subset $U \subset X$

$$\mathrm{ParHom}\,(E,F)(U) = \{f \in \mathrm{Hom}\,(E,F)(U) \text{ such that } f \text{ is strongly parabolic}\}.$$

Lemma 1.1.9. *Let E and F parabolic bundles over X with marked points on D. Then, $\mathrm{ParHom}\,(E,F)$ and $\mathrm{SParHom}\,(E,F)$ are vector bundles over X with degrees*

$$\begin{aligned} \deg(\mathrm{ParHom}\,(E,F)) &= \mathrm{rk}\,(E)\deg(F) - \mathrm{rk}\,(F)\deg(E) \\ &+ \sum_{x\in D}(\dim(\mathrm{ParHom}\,(E_x,F_x) - \mathrm{rk}\,(E)\,\mathrm{rk}\,(F)) \qquad (1.2) \\ \deg(\mathrm{SParHom}\,(E,F)) &= \mathrm{rk}\,(E)\deg(F) - \mathrm{rk}\,(F)\deg(E) \\ &+ \sum_{x\in D}(\dim(\mathrm{SParHom}\,(E_x,F_x) - \mathrm{rk}\,(E)\,\mathrm{rk}\,(F))) \qquad (1.3) \end{aligned}$$

Proof. It follows from the existence of torsion sheaves T_1 and T_2 supported on D such that

$$0 \to \mathrm{ParHom}\,(E,F) \to \mathrm{Hom}\,(E,F) \to T_1 = \bigoplus_{x\in D} \frac{\mathrm{Hom}\,(E_x,F_x)}{\mathrm{ParHom}\,(E_x,F_x)} \to 0. \qquad (1.4)$$

and

$$0 \to \mathrm{SParHom}\,(E,F) \to \mathrm{Hom}\,(E,F) \to T_2 = \bigoplus_{x\in D} \frac{\mathrm{Hom}\,(E_x,F_x)}{\mathrm{SParHom}\,(E_x,F_x)} \to 0. \qquad (1.5)$$

□

Lemma 1.1.10. *Let L and L' be two parabolic line bundles with sistems of weights α and α' respectively. Then,*

$$\mathrm{SParHom}\,(L, L'(D)) \cong \mathrm{Hom}\,(L, L'(S))$$

where $S = \{x \in D \text{ such that } \alpha(x) < \alpha'(x)\}$.

Proof. By definition a strongly parabolic homomorphism $f : L_1 \to L_2(D)$ satisfies $\mathrm{Res}_p f = 0$ if and only if $\alpha(x) \geq \alpha'(x)$. From this the result is clear. □

Definition 1.1.11. Let E be a parabolic bundle with system of weights α and le L be a parabolic line bundle with system of weights η. The parabolic tensor $E \otimes^p L$ is, as a bundle, the kernel of

$$E \otimes L(D) \twoheadrightarrow \bigoplus_{x\in D}((E_x/E_{x,i_x}) \otimes L(D)_x)$$

where $i_x = \min\{r(x)+1, i|\alpha_i(x)+\eta(x) \geq 1\}$, $x \in D$. The weights of $E \otimes^p L$ are

$$\alpha_{i_x}(x)+\eta(x)-1 < \cdots < \alpha_{rx}(x)+\eta(x)-1 < \alpha_1(x)+\eta(x) < \cdots < \alpha_{i_x-1}+\eta(x),$$

with multiplicities $m_{i_x}(x), \ldots, m_{r(x)}(x), m_1(x), \ldots, m_{i_x-1}(x)$.

Lemma 1.1.12. *Let E a parabolic bundle and L a parabolic line bundle. Then,*

$$\operatorname{pardeg}(E \otimes^p L) = \operatorname{pardeg}(E) + \operatorname{rk}(E)\operatorname{pardeg}(L).$$

Proof. Observe that

$$\deg(E \otimes^p L) = \deg(E) + \operatorname{rk}(E)\deg(L) + \sum_{x \in D} \dim E_{x,i_x},$$

hence,

$$\operatorname{pardeg}(E \otimes^p L) = \deg(E) + \operatorname{rk}(E)\deg(L) + \sum_{x\in D} \dim E_{x,i_x} + \sum_{x \in D}\sum_i (\alpha_i(x)+\eta(x)) - \dim E_{x,i_x}.$$

□

Lemma 1.1.13. *Let E be a parabolic bundle and L a parabolic line bundle. Then*

$$\operatorname{ParHom}(E, L) \cong E^* \otimes^p L.$$

□

Let E and F be parabolic bundles over X with full flags on each marked point $p \in D$, and weight types α and β respectively. Let z be a local holomorphic coordinate on X near a marked point $x \in D$, such that x corresponds to $z = 0$. Let $\{e_i\}$ and $\{f_j\}$ be bases for E_x and F_x adapted to the filtrations of E and F at x, that is $E_{x,i} =< e_i, \ldots, e_r >$ and $F_{x,j} =< f_j, \ldots, f_{r'} >$ (where r and r' are the dimensions of the vector spaces E_x and F_x).

Using this notation the stalk $\operatorname{Hom}(E,F)_{<x>} = \mathbb{C}[[z]][a_{ij}]$ where (a_{ij}) are coordinate functions on $r \times r'$ matrices.

Let

$$b_{ij} = \begin{cases} a_{ij} \text{ if } \alpha_i \leq \beta j \\ z a_{ij} \text{ if } \alpha_i > \beta_j \end{cases}.$$

Hence, with respect to the chosen bases, $\operatorname{ParHom}(E,F)_x = \mathbb{C}[[z]][b_{ij}]$.

A parabolic homomorphism from E to F can be written locally as

$$f = f_0 + f_1 z + f_2 z^2 + \cdots$$

where $f_0 \in \operatorname{ParHom}(E_x, F_x)$, and $f_i \in \operatorname{Hom}(E_x, F_x)$ for $i \geq 1$, and z is the local coordinate for a trivialisation around x with $z(x) = 0$.

The space $\operatorname{ParHom}(E_x, F_x)$ is a vector space consisting of matrices whose non-zero entries are determined by the weights on E and F over x.

Definition 1.1.14. A *generic* element of $\operatorname{ParHom}(E_x, F_x)$ is an element of the open dense orbit of the action of the group of parabolic automorphisms of this vector spaces.

Example 1.1.15. Consider the parabolic vector spaces given in Example 1.1.7 by the fibres over x of E and F, a parabolic homomorphism from E_x to F_x is given by a matrix of the following form

$$\begin{pmatrix} 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix},$$

where $a_{i,j} \in \mathbb{C}$.

The space $\operatorname{ParHom}(E_x, F_x)$ is a vector space and it is in particular an affine variety. Considering the action of the group $\operatorname{ParAut}(E_x) \times \operatorname{ParAut}(F_x)$ (lower triangular matrices changing the bases of E_x and F_x) on this space we find that there is one open dense orbit.

In this example, the generic parabolic homomorphism from E_x to F_x has the following matrix form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where the non-zero entries of the matrix are disposed such that there is no more non-zero entries on its column or file, and the corners of the broken line are filled by non-zero entries.

Lemma 1.1.16. *Let X be a compact Riemann surface with a finite set of marked points with associated effective divisor D, and let E and F be parabolic vector bundles on X. Let $x \in D$. Then, there is an exact sequence*

$$0 \to \frac{\operatorname{Hom}(E_x, F_x)}{\operatorname{ParHom}(E_x, F_x)} \otimes \mathcal{O}(-x) \to \operatorname{ParHom}(E, F)_x \to \operatorname{ParHom}(E_x, F_x) \to 0.$$

Proof. From the exact sequence in (1.4), tensoring it by the skyscraper sheaf $\mathbb{C}(x)$, we get

$$
\begin{aligned}
0 \ \to \ & \mathrm{Tor}\,\Big(\frac{\mathrm{Hom}\,(E_x, F_x)}{\mathrm{ParHom}\,(E_x, F_x)}, \mathbb{C}(x)\Big) \to \mathrm{ParHom}\,(E, F)_x \to \mathrm{Hom}\,(E, F)_x \to \\
\to \ & \frac{\mathrm{Hom}\,(E_x, F_x)}{\mathrm{ParHom}\,(E_x, F_x)} \to 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
0 \ \to \ & \frac{\mathrm{Hom}\,(E_x, F_x)}{\mathrm{ParHom}\,(E_x, F_x)} \otimes \mathcal{O}(-x) \to \mathrm{ParHom}\,(E, F)_x \to \mathrm{Hom}\,(E_x, F_x) \to \\
\to \ & \frac{\mathrm{Hom}\,(E_x, F_x)}{\mathrm{ParHom}\,(E_x, F_x)} \to 0,
\end{aligned}
$$

which gives

$$
0 \to \frac{\mathrm{Hom}\,(E_x, F_x)}{\mathrm{ParHom}\,(E_x, F_x)} \otimes \mathcal{O}(-x) \to \mathrm{ParHom}\,(E, F)_x \to \mathrm{ParHom}\,(E_x, F_x) \to 0,
$$

where the first map is multiplication by z, and the second map is the restriction to x obtained by setting $z = 0$. $\square$

As an example see that $f \in \mathrm{ParHom}\,(E, F)_x$ has matrix form

$$
\left(\begin{array}{ccc}
a_{1,1}z & a_{1,2}z & a_{1,3}z \\
\hline
a_{2,1} & a_{2,2} & a_{2,3}z \\
a_{3,1} & a_{3,2} & a_{3,3}z \\
a_{4,1} & a_{4,2} & a_{4,3}
\end{array}\right)
$$

The Serre duality theorem was generalised by Yokogawa.

Theorem 1.1.17 (K.Yokogawa, [Y1]). *Let X be a compact Riemann surface with an effective divisor D as above. For parabolic bundles E and F over X there is a natural isomorphism*

$$
H^i(\mathrm{ParHom}\,(E, F) \otimes K(D)) \xrightarrow{\cong} H^{1-i}(\mathrm{SParHom}\,(F, E))^*.
$$

Definition 1.1.18. Let E be a parabolic bundle with system of weights α, and let $E' \subset E$ be a holomorphic subbundle of E. We said that E' is provided with the *induced parabolic structure* if we give to E' the following parabolic structure. Denote $r' = \mathrm{rk}\,(E')$ and chose i the smallest integer such that $\mathrm{rk}\,(E_{i,x}) \leq r'$ then,

$$
E'_{x,j} = E' \cap E_{x,i+j} \quad \text{and } \alpha_j = \alpha_{i+j}.
$$

Definition 1.1.19. Let E be a parabolic bundle and E' is a holomorphic subbundle of E provided with the induced parabolic structure. A *parabolic quotient* is defined as a parabolic bundle E'' satisfying:

(i) for all E''_j from the filtration on E'' there is a term E_i on the filtration on E such that $g(E_i) = E''_j$.

(ii) if i is the greatest integer for which (i) is true then, $\alpha_i = \beta_j$.

Definition 1.1.20. Let E, E', E'' be parabolic vector bundles over X. We say that E is *the direct sum of E' and E''* if $E = E' \oplus E''$ as holomorphic bundles and the parabolic structure on E has the following system of weights

$$\{\alpha_{i,x}\}_{i=1,n(x)} = \{\alpha'_j\}_{j=1,n'(x)} \cup \{\alpha''_k\}_{k=1,n''(x)}$$

and the corresponding filtration is such that

$$E_{x,i} = E'_{x,j} \oplus E''_{x,k}$$

where j (resp. k) is the smallest integer such that $\alpha_{x,i} \leq \alpha'_{x,j}$ (resp. $\alpha_{x,i} \leq \alpha''_{x,k}$). We denote by $\alpha' \cup \alpha''$ the system of weights on E.

Remark 1.1.21. The system of weights of E splits in the system of weights of E' and E'' and, conversely, the two systems of weights of E' and E'' form a partition for the system of weights of E.

Lemma 1.1.22. *Let E be a parabolic bundle such that $E = E' \oplus E''$. Then*

$$\text{pardeg}\,(E) = \text{pardeg}\,(E') + \text{pardeg}\,(E'').$$

□

Definition 1.1.23. Let f be a parabolic homomorphism from E to F we define the parabolic kernel of f as

$$\ker(f) = \bigcup_{x \in X} f_x,$$

provided with the induced parabolic structure from E over each $x \in D$.

We denote $\mathrm{Im}\,(f) = \bigcup_{x\in X} f_x$ but note that there is not a canonical parabolic structure for $\mathrm{Im}\,(f)$. We can give it the induced parabolic structure from F. In this case $\mathrm{Im}\,(f)$ becomes a parabolic subbundle of F. But, we can also give it the parabolic structure making

$$0 \to \ker(f) \to E \to \mathrm{Im}\,(f) \to 0$$

a short exact sequence that is, the filtration is given by the filtration on F but the system of weights consists of the weights that of E that does not appear in the system of weights of parabolic kernel.

1.2 Extensions of parabolic bundles.

Definition 1.2.1. Let E' and E'' be two parabolic bundles over X. An extension of E'' by E' is given by the following short exact sequence,

$$0 \to E' \to E \to E'' \to 0,$$

such that the parabolic structure on E induces the parabolic structures of E' and E''. We denote the group of parabolic extensions of E'' by E' by $\mathrm{ParExt}\,(E'', E')$.

Lemma 1.2.2. *Let E' and E'' be two parabolic bundles over X. Then*

$$\mathrm{ParExt}\,(E'', E') \cong H^1(\mathrm{ParHom}\,(E'', E')).$$

Lemma 1.2.3. *Let E and F be two parabolic bundles. Then*

$$\mathrm{pardeg}\,(E \otimes^p F) = \mathrm{rk}\,(F)\,\mathrm{pardeg}\,(E) + \mathrm{rk}\,(E)\,\mathrm{pardeg}\,(F).$$

Proof. This is a consequence of Lemmas 1.1.12 and 1.1.22. □

Lemma 1.2.4. *Let E and F be two parabolic bundles. Then*

$$\mathrm{ParHom}\,(E, F) \cong E^* \otimes^p F.$$

Proof. This is a consequence of Lemmas 1.1.13 and 1.1.22. □

1.3 Orbifold bundles and parabolic bundles.

An interesting example of parabolic bundle is an orbifold bundle. The relationship between parabolic bundles and orbifold bundles (also called V-bundles) has been studied by several authors (see [Bi, Bo, NaSt]). This relationship is obtained by the push forward construction, we are about to describe it, but first, following Thurston [T], we define what an orbifold surface and an orbifold bundle is.

Definition 1.3.1. An *orbifold Riemann surface* O is a compact, connected Riemann surface, together with a finite number of marked points with, at each marked point, an associated order of isotropy m (an integer grater than one).

Note that, although every point on the surface has a neighbourhood modelled on D^2 (open unit disk in $\mathbb{C}$), a neighbourhood of a marked point has the form $D^2/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts on $\mathbb{C}$ in the standard way as the m-th roots of the unity. Let σ denote the standard representation of $\mathbb{Z}_m$, with generator $\zeta = e^{\frac{2\pi\sqrt{-1}}{m}}$. At a marked point x is locally D^2/σ.

Remark 1.3.2. Given a surface with a branched covering we naturally consider it to be an orbifold surface by marking a branch point with isotropy given by the ramification index.

Definition 1.3.3. A *complex orbifold bundle* of rank r is a vector bundle over O such that for a local trivialisation around each marked point of O it has the form $E|_{D^2/\sigma} \to (D^2 \times \mathbb{C}^r)/(\sigma \times \rho)$, where ρ is an isotropy representation $\rho : \mathbb{Z}_m \to \mathrm{GL}(r, \mathbb{C})$.

We can always choose coordinates in an orbifold bundle which respect the orbifold structure, that is, if the isotropy representation is $\rho : \mathbb{Z}_m \to \mathrm{GL}(r, \mathbb{C})$ then we can choose coordinates so that ρ decomposes as $\rho = \sigma^{k_1} \oplus \sigma^{k_2} \oplus \cdots \oplus \sigma^{k_r}$, where for j=1,r, k_j is an integer with $0 \leq k_j < m$ and the k_j are increasing.

The following proposition proved by Boden [Bo] and Furuta and Steer [FuSt] gives us the relationship between orbifold bundles and parabolic bundles.

Proposition 1.3.4. *Let O be an orbifold surface and $|O|$ the underlying Riemann surface. Given an orbifold bundle V over O, there is a natural parabolic bundle E over $|O|$. Where the parabolic weights are rational numbers of the form $k_{x,j}/m(x)$ where $m(x)$ is the order of the marked point x in O and $k_{x,i}$ is given by the isotropy representation. Here by natural we mean that given a holomorphic homomorphism of orbifold bundles $\varphi : V_1 \to V_2$, there is an associated parabolic homomorphism $\tilde{\varphi} : E_1 \to E_2$.*

$\square$

Remark 1.3.5. Not every parabolic bundle can be pulled back to an orbifold bundle, since it must have rational weights.

Nasatyr and Steer in [NaSt] proved that this correspondence between parabolic bundles with rational weights and orbifold bundles induce a correspondence between their moduli spaces.

1.4 Parabolic bundles and representations of the fundamental group.

Let X be a compact Riemann surface of genus $g \geq 0$. Let $\tilde{X}$ be the universal covering space of X.

Let V be finite dimensional complex vector space with an hermitian metric, and let $\rho : \pi_1(X) \to U(V)$ be a unitary representation. There is a vector bundle over X associated to ρ. Taking the representation above we consider the action of $\pi_1(X)$ on $\tilde{X} \times V$ given by,

$$\pi_1 \times (\tilde{X} \times V) \rightarrow \tilde{X} \times V \tag{1.6}$$

$$(g, (z, v)) \mapsto (zg, \rho(g^{-1}v)) \tag{1.7}$$

The quotient $E_\rho = \tilde{X} \times V/\pi_1(X)$ give us a vector bundle over X of rank $\mathrm{rk}\,(E_\rho) = \dim(V)$.

A theorem of Narasimhan and Seshadri [NS] states that the bundle E_ρ is stable if and only if ρ is irreducible. Moreover, it also says that the map $\rho \mapsto E_\rho$ defines a one-to-one correspondence between isomorphism classes of irreducible and unitary representations of $\pi_1(X)$ and isomorphism classes of stable bundles over X with $\deg(E) = 0$.

In the parabolic case, there is an analogous correspondence given by Mehta and Seshadri in [MS].

Let X be Riemann surface of genus $g \geq 0$ and let $\{x_i\}_{i=1}^s \subset X$ be a finite subset of points in X. Lets consider $\tilde{X}$ the universal covering space of $X \backslash \{x_1, \ldots, x_s\}$ and $\pi_1(X \backslash \{x_1, \ldots, x_s\})$ its fundamental group. Adding the set P, of parabolic points of $X \setminus \{x_1, \ldots, x_s\}$, to $\tilde{X}$ we get $\tilde{X}' = \tilde{X} \cup P$ such that $\tilde{X}'/\pi_1(X \setminus \{x_1, \ldots, x_s\}) = X$.

A unitary representation $\rho : \pi_1(X \setminus \{x_1, \ldots, x_s\}) \to U(V)$ defines a vector bundle E_ρ^0 over $X \setminus \{x_1, \ldots, x_s\}$. Considering appropriate bounded functions on the marked points of X we can extend E_ρ^0 to a vector bundle E_ρ over X

Let I be the isotropy subgroup of $\pi_1(X \setminus \{x_1, \dots, x_s\})$ on a parabolic point $p \in P$, that is. the subgroup that fixes p under the action of $\pi_1(X \setminus \{x_1, \dots, x_s\})$ on $\tilde{X}'$. This is isomorphic to $\mathbb{Z}$, therefore choosing a suitable orthonormal basis of V the image under ρ of the generator ξ of I will be

$$\rho(\xi) = \begin{pmatrix} exp(\sqrt{-1}\pi\tilde{\alpha}_1) & & 0 \\ & \ddots & \\ 0 & & exp(\sqrt{-1}\pi\tilde{\alpha}_n) \end{pmatrix},$$

where n is the complex dimension of V, and $0 \leq \tilde{\alpha}_1 \leq \cdots \leq \tilde{\alpha}_r < 1$.

Take $\alpha_1 = \tilde{\alpha}_1$ and $\alpha_i = \min\{\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_r\} \setminus \{\alpha_1, \dots, \alpha_{i-1}\}\}$ we get $0 \leq \alpha_1 < \cdots < \alpha_r < 1$.

Let $E_i =< e_{k(i)}, \dots, e_n >$ where $k(i)$ is such that $\beta_{k(i)}) = \alpha_i$ we get a filtration on E_ρ

$$E_\rho\big|_{x_j} = E_1 \supset E_2 \supset \cdots \supset E_{r(x_j)} \supset \{0\}.$$

The bundle E_ρ with the filtration and system of weights described above is a parabolic vector bundle.

It is possible to define suitable metrics and connections on this bundle such that we can define parabolic degree and parabolic stability notions. Moreover, there is a parabolic counterpart of the Narasimhan Seshadri theorem given by Mehta and Seshadri in [MS].

Theorem 1.4.1 ([MS]). *Let ρ be an irreducible representation of $\pi_1(X - D)$, and E_ρ the corresponding parabolic bundle described above. If ρ is irreducible then E_ρ is parabolically stable with parabolic degree zero. Conversely, if E is a parabolic bundle over X with marked points on D, with null parabolic degree and parabolically stable, then, there is a representation $\rho : \pi_1(X - D) \to U(r)$ (with $r = \mathrm{rk}\,(E)$) such that $E_\rho \cong E$.*

Mehta and Seshadri's theorem was proved using gauge theory by Biquard in [B]. Next, we summarise his results from [B] and state the gauge theoretic formulation of the theorem.

Definition 1.4.2. $\{e_i\}$ is a basis of E at $x \in D$ compatible with the filtration if it is a local basis of E around x such that

$$E^i_x =< e_{r-d+1}, \dots, e_r >$$

where r is the rank of E and $d = \dim E^i_x$.

Definition 1.4.3. Given $p > 1$ a local basis $\{e_i\}$ of sections of E in a neighbourhood of $x \in S$ is said to be

i) adapted to E if $e_i = f_i/|z|^{\alpha_i}$ where f_i is a local basis of E at x compatible with the filtration;

ii) adapted to (E, h) with h hermitian metric, if it is adapted to E and h-orthonormal.

Definition 1.4.4. A hermitian metric h over $E|_{X-\{x\}}$ is called *adapted to the parabolic structure* if there is a basis $\{e_i\}$ adapted to (E, h).

Remark 1.4.5. Observe that it is possible always to find an adapted metric for a given parabolic bundle E. But, for a parabolic bundle E provided with an adapted hermitian metric is not always possible to find an adapted basis which is also orthonormal with respect to the metric.

Given a parabolic bundle E over X with set of marked points D. We can obtain a adapted metric to E on $x \in S$ in the following way.

First, let z be a local coordinate for x, and let $\{e_i\}$ a local basis compatible with the filtration. We have the metric

$$h = \begin{pmatrix} |z|^{2\tilde{\alpha}_1} & & \\ & \ddots & \\ & & |z^{2\tilde{\alpha}_r}| \end{pmatrix}$$

although the local basis $\{e_i\}$ is not orthonormal so we can modify it to a basis $\{f_i\}$ defined by

$$\{f_i\}, \quad f_i = \frac{e_i}{|z|^{\tilde{\alpha}_i}}$$

such that $\{f_i\}$ is a a basis adapted to (E, h)

$$h(f_i, f_i) = h\Big(\frac{e_i}{|z|^{\tilde{\alpha}_i}}, \frac{e_i}{|z|^{\tilde{\alpha}_i}}\Big) = 1.$$

Proposition 1.4.6 (O. Biquard, [B]). *Let h be an adapted metric for a parabolic bundle E and, let E' be a parabolic subbundle of E. Then $h|_{E'}$ is adapted to E'.*

Theorem 1.4.7 (O. Biquard, [B]). *Let X and S be as before. There is an equivalence of categories between*

i) parabolic bundles over X with parabolic structures on D, and

ii) holomorphic hermitian bundles on $X - S$ with L^p curvature for some $p > 1$, where the morphisms are holomorphic bounded morphisms on $X - S$.

One has the following result.

Proposition 1.4.8 (O. Biquard, [B]). *Let E be a holomorphic parabolic bundle on the compact Riemann surface X and, let h be an adapted metric on $X - S$. Then*

$$-\frac{1}{2\pi i}\int_{X-S} Tr(F_h) = \deg(E) + \sum_{\substack{1\le i\le n(x)\\ x\in S}} \alpha_{x,i}.$$

This is a parabolic version of the Chern-Weil theorem.

Definition 1.4.9. Let E be a parabolic bundle with system of weights α. The *parabolic degree of* E is defined as

$$\text{pardeg } E := \deg(E) + \sum_{\substack{1\le i\le n(x)\\ x\in S}} \alpha_{x,i}.$$

Definition 1.4.10. Let E be a parabolic bundle over X. The *parabolic slope of* E is defined as

$$\text{par}\mu\,(E) := \frac{\text{pardeg}\,(E)}{\text{rk}\,(E)}$$

where $\text{rk}\,(E)$ is the rank of E.

Definition 1.4.11. A parabolic bundle E is *parabolically (semi)stable* if and only if for all subbundles E' of E their parabolic slopes are such that

$$\text{par}\mu\,(E')(\le) < \text{par}\mu\,(E).$$

Finally we give a gauge-theoretic formulation of the Mehta and Seshadri theorem.

Theorem 1.4.12. *Let E be a parabolic bundle over a Riemann surface X with set of marked points $D = x_1 + \cdots + x_s$. Let h be an hermitian adapted metric to the parabolic structure on E. Then, E is parabolically stable if and only if there is a singular (with poles of order one) unitary connection A such that*

$$F_A = -2\pi\sqrt{-1}\,\text{par}\mu\,(E)\,\text{Id}_E\,.$$

This connection is unique up to the action of the gauge group of E.

Chapter 2

Parabolic triples.

2.1 Definitions.

Let X be a compact Riemann surface of genus $g \geq 0$ together with $\{x_i\}_{i=1}^{s} \subset X$ a finite set of marked points. Let $D = x_1 + \ldots + x_s$ be the corresponding effective divisor.

Definition 2.1.1. A *parabolic triple* is a triple $T = (E_1, E_2, \phi)$ consisting of two parabolic bundles over X, E_1 and E_2, and a strongly parabolic homomorphism $\phi : E_2 \to E_1(D)$, i.e. an element $\phi \in H^0(\mathrm{SParHom}\,(E_2, E_1(D)))$.

Definition 2.1.2. A homomorphism of parabolic triples f from $T = (E_1, E_2, \phi)$ to $T' = (E_1', E_2', \phi')$ is a pair of parabolic homomorphisms $f = (f_1, f_2)$ making the following diagram commutative

$$\begin{array}{ccc} E_2 & \xrightarrow{\phi} & E_1(D) \\ \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f_1} \\ E_2' & \xrightarrow{\phi'} & E_1'(D) \end{array}$$

Definition 2.1.3. A parabolic triple $T' = (E_1', E_2', \phi')$ is a *parabolic subtriple* of $T = (E_1, E_2, \phi)$ if $E_i' \subset E_i$ are parabolic subbundles for $i = 1, 2$ and $\phi'(E_2') \subset E_1'(D)$ where ϕ' is the restriction of ϕ to E_2'.

Definition 2.1.4. Let σ be a real number, we define the parabolic σ*-slope* of T by

$$\mathrm{par}\mu_{\sigma}(T) = \frac{\mathrm{pardeg}\,(E_1) + \mathrm{pardeg}\,(E_2)}{\mathrm{rk}\,(E_1) + \mathrm{rk}\,(E_2)} + \sigma \frac{\mathrm{rk}\,(E_2)}{\mathrm{rk}\,(E_1) + \mathrm{rk}\,(E_2)}.$$

Definition 2.1.5. A parabolic triple is called *σ-stable* if $\mathrm{par}\mu_\sigma(T') < \mathrm{par}\mu_\sigma(T)$ for all proper parabolic subtriples $T' \subset T$. It is called *σ-semistable* by requiring the weaker inequality instead of the strict inequality and, T is defined to be *σ-polystable* if T decomposes as a direct sum of parabolic triples of the same σ-slope. The parabolic triple is said to be σ-unstable if $\mathrm{par}\mu_\sigma(T') > \mathrm{par}\mu(T)$ for all proper substriple $T' \subset T$.

Definition 2.1.6. Given a parabolic triple $T = (E_1, E_2, \phi)$, its *dual triple* is a triple $T^* = (E_2^*, E_1^*, \phi^*)$ where E_i^* is the parabolic dual of E_i and ϕ^* is the transpose of ϕ.

Proposition 2.1.7. *The σ-(semi)stability of T is equivalent to the σ-(semi)stability of T^*. The map $T \mapsto T^*$ defines an isomorphism of moduli spaces.*

Proof. For the first statement we only have to recall that $\mathrm{pardeg}\,(E^*) = -\,\mathrm{pardeg}\,(E)$, hence

$$\mathrm{par}\mu_\sigma(T^*) = -\frac{\mathrm{pardeg}\,(E_1) + \mathrm{pardeg}\,(E_2)}{r_1 + r_2} + \left(1 - \frac{r_2}{r_1 + r_2}\right).$$

Therefore, if $\mathrm{par}\mu_\sigma(T') < \mathrm{par}\mu_\sigma(T)$ then $\mathrm{par}\mu_\sigma(T'^*) > \mathrm{par}\mu_\sigma(T^*)$. Observe that, each subtriple T' of T becomes a quotient under the map $T \mapsto T^*$ and conversely.

If we have an isomorphism $f = (f_1, f_2)$ between two triples T and T' we get an isomorphism between T^* and T'^* by considering $f^* = (f_1^*, f_2^*)$ where $f_i^* : E_i^* \to E_i'^*$, $\zeta_i \mapsto \zeta_i \circ f_i$, with $i = 1, 2$. And we know that f^* commutes with ϕ^* and ϕ'^* since f commutes with ϕ and ϕ'. Hence, the map $T \mapsto T^*$ descends to isomorphism classes of parabolic triples.

□

Note that one can define the direct sum of parabolic triples T' and T'', and similarly any other operation. This consist of first do the direct sum of the parabolic bundles in the triples $E_1' \oplus E_1''$, $E_2' \oplus E_2''$, and then, we sum the homomorphism ϕ' and ϕ'' such that the corresponding diagrams commute. That is, $T = T' \oplus T''$ is the triple $T = (E_1, E_2, \phi)$ where $E_1 = E_1' \oplus E_1''$, $E_2 = E_2' \oplus E_2''$ and $\phi = \phi' \oplus \phi''$.

Definition 2.1.8. Let $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ be the moduli space of isomorphism classes of σ-stable triples with fixed weight topological type $(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$; that is, with sistems of weights (α^1, α^2), ranks $r_1 = \mathrm{rk}\,(E_1)$, $r_2 = \mathrm{rk}\,(E_2)$, and degrees $d_1 = \deg(E_1)$, $d_2 = \deg(E_2)$ for the parabolic bundles E_1, E_2 in the triple.

Using the dimensional reduction construction given in [BG], one can show that the moduli space $\mathcal{N}_\sigma$ is a subvariety of a certain moduli space of parabolic sheaves on $X \times \mathbb{P}^1$. Such moduli spaces have been constructed by Maruyama and Yokogawa [MY] in arbitrary dimensions using Geometric Invariant Theory methods.

Proposition 2.1.9. *A necessary condition for $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2, \alpha^1, \alpha^2)$ to be non-empty is that*

$$\begin{array}{ll} \sigma_m < \sigma < \sigma_M & \text{if} \quad r_1 \neq r_2 \\ \sigma_m < \sigma & \text{if} \quad r_1 = r_2 \end{array}$$

where

$$\begin{aligned} \sigma_m &= \operatorname{par}\mu\,(E_1) - \operatorname{par}\mu\,(E_2) \\ \sigma_M &= \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\operatorname{par}\mu\,(E_1) - \operatorname{par}\mu\,(E_2)) + s\frac{r_1 + r_2}{|r_1 - r_2|} \quad \textit{if}\, r_1 \neq r_2. \end{aligned}$$

Proof. Let $r_1 \neq r_2$ and let $T = (E_1, E_2, \phi)$ be a stable parabolic triple in $\mathcal{N}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$. Let N be the kernel of ϕ and I be the image of ϕ. We consider the subtriples $T_1 = (0, N, \phi)$ and $T_2 = (I(-D), E_2, \phi)$. At least one of them is proper since $r_1 \neq r_2$ and ϕ can not be an isomorphism. Suppose both are proper subtriples (if only one is, then the reasoning is analogous). Since T is a stable triple we have that

$$\begin{aligned} \operatorname{par}\mu_\sigma(T_1) &< \operatorname{par}\mu_\sigma(T) \\ \operatorname{par}\mu_\sigma(T_2) &< \operatorname{par}\mu_\sigma(T). \end{aligned} \tag{2.1}$$

Note that $d_2 = d_N + d_I$ and $r_2 = r_N + r_I$ where $d_N = \deg(N)$, $d_I = \deg(I)$, $r_N = \operatorname{rk}(N)$ and $r_I = \operatorname{rk}(I)$. Hence, combining the inequalities in (2.1) we obtain the upper bound for σ. □

2.2 Extensions and deformations of parabolic triples.

We collect in this section some results from [GGM] about extensions and deformations of parabolic triples.

Let $T' = (E_1', E_2', \phi')$ and $T'' = (E_1'', E_2'', \phi'')$ be two parabolic triples. Let $\operatorname{Hom}(T'', T')$ be the vector space of homomorphisms from T'' to T', and $\operatorname{Ext}^1(T'', T')$ be the vector space of extensions of the form

$$0 \to T' \to T \to T'' \to 0,$$

that is, a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_2' & \longrightarrow & E_2 & \longrightarrow & E_2'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & E_1'(D) & \longrightarrow & E_1(D) & \longrightarrow & E_1''(D) & \longrightarrow & 0. \end{array}$$

To study extensions of parabolic triples, we consider the hypercohomology of the complex of sheaves defined by

$$\begin{aligned} C^\bullet(T'',T') : \mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2') &\rightarrow \mathrm{SParHom}\,(E_2'',E_1'(D)) \\ (\psi_1,\psi_2) &\mapsto \phi'\psi_2 - \psi_1\phi''. \end{aligned} \tag{2.2}$$

Proposition 2.2.1. *There are natural isomorphisms*

$$\begin{aligned} \mathrm{Hom}\,(T'',T') &\cong \mathbb{H}^0(C^\bullet(T'',T')) \\ \mathrm{Ext}^1(T'',T') &\cong \mathbb{H}^1(C^\bullet(T'',T')), \end{aligned}$$

and a long exact sequence:

$$\begin{aligned} 0 &\rightarrow \mathbb{H}^0 \rightarrow H^0(\mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2')) \rightarrow H^0(\mathrm{SParHom}\,(E_2'',E_1'(D))) \\ &\rightarrow \mathbb{H}^1 \rightarrow H^1(\mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2')) \rightarrow H^1(\mathrm{SParHom}\,(E_2'',E_1'(D))) \\ &\rightarrow \mathbb{H}^2 \rightarrow 0. \end{aligned} \tag{2.3}$$

□

Let $h^i(T'',T') = \dim \mathbb{H}^i(C^\bullet(T'',T'))$ be the dimension of the i-hypercoholomogy group of the complex $C^\bullet(T'',T')$ and let $\chi(T'',T')$ be the alternating sum

$$\chi(T'',T') = h^0(T'',T') - h^1(T'',T') + h^2(T'',T'). \tag{2.4}$$

Proposition 2.2.2. *Let T' and T'' be parabolic triples. Then*

$$\chi(T'',T') = \chi(\mathrm{ParHom}\,(E_1'',E_1')) + \chi(\mathrm{ParHom}\,(E_2'',E_2')) - \chi(\mathrm{SParHom}\,(E_2'',E_1'(D))).$$

Proof. Take the euler characteristic in (2.3). □

Corollary 2.2.3. *For any extension $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ of parabolic triples we have that*

$$\chi(T,T) = \chi(T',T') + \chi(T'',T'') + \chi(T'',T') + \chi(T',T'').$$

□

Proposition 2.2.4. *Suppose that T' and T'' are σ-semistable.*

(i) If $\mathrm{par}\mu_\sigma(T') < \mathrm{par}\mu_\sigma(T'')$ *then* $\mathbb{H}^0(C^\bullet(T'', T')) \cong 0$.

(ii) If $\mathrm{par}\mu\,(T') = \mathrm{par}\mu\,(T'')$ *and T', T'' are σ-stables, then*

$$\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} \mathbb{C} \text{ if } T' \cong T'' \\ 0 \text{ if } T' \ncong T''. \end{cases} \tag{2.5}$$

□

We conclude with some results on the moduli space of parabolic triples taken from [GGM].

Theorem 2.2.5. *Let $T = (E_1, E_2, \phi)$ be a σ-stable parabolic triple.*

(i) The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(C^\bullet(T, T))$.

(ii) If $\mathbb{H}^2(C^\bullet(T, T)) = 0$, then the moduli space of σ-stable parabolic triples is smooth in a neighbourhood of the point defined by T.

(iii) $\mathbb{H}^2(C^\bullet(T, T)) = 0$ *if and only if the homomorphism*

$$H^1(\mathrm{ParEnd}\,(E_1)) \oplus H^1(\mathrm{ParEnd}\,(E_2)) \to H^1(\mathrm{SParHom}\,(E_2, E_1(D)))$$

is surjective.

(iv) At a smooth point $T \in \mathcal{N}_\sigma$ the dimension of the moduli space of σ-stable parabolic triples is

$$\begin{aligned} \dim \mathcal{N}_\sigma =& h^1(T, T) = 1 - \chi(T, T) \\ =& \chi(\mathrm{ParEnd}\,(E_1)) + \chi(\mathrm{ParEnd}\,(E_2)) - \chi(\mathrm{SParHom}\,(E_2, E_1(D))) \end{aligned}$$

(v) If ϕ is injective or surjective then T defines a smooth point in the moduli space.

□

2.3 Critical values for the parameter σ.

A parabolic triple $T = (E_1, E_2, \phi)$ of type $(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ is *strictly σ-semistable* if and only if there is a proper subtriple T' such that $\mathrm{par}\mu_\sigma(T) = \mathrm{par}\mu_\sigma(T')$, that is,

$$\mathrm{par}\mu(T') + \sigma \frac{r_2'}{r_1' + r_2'} = \mathrm{par}\mu(T) + \sigma \frac{r_2}{r_1 + r_2}. \tag{2.6}$$

There are two ways in which this can happen. The first one is by the existence of a parabolic subtriple such that

$$\frac{r_2'}{r_1' + r_2'} = \frac{r_2}{r_1 + r_2},$$

and hence

$$\mathrm{par}\mu(T') = \mathrm{par}\mu(T).$$

In this case T is strictly σ-semistable for all σ (or at least in an interval for σ) and it is called *σ-independent semistable*. The other way in which strict σ-semistability can happen is if equality holds for (2.6) but with

$$\frac{r_2'}{r_1' + r_2'} \neq \frac{r_2}{r_1 + r_2}.$$

Definition 2.3.1. The values for σ such that for some T there is a subtriple T' such that

$$\frac{r_2'}{r_1' + r_2'} \neq \frac{r_2}{r_1 + r_2}$$

are called *critical values.*

Proposition 2.3.2. *(i) The critical values of σ form a discrete subset of $[\sigma_m, \sigma_M]$. Here we adopt the convention that $\sigma_M = \infty$ when $r_1 = r_2$.*

(ii) The stability criteria for two values of σ between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.

(iii) For parabolic triples consisting of parabolic bundles with generic weights, $\sigma = 2g - 2$ is not a critical value. Where by generic weights for a triple T we mean that σ-semistability implies σ-stability.

2.4 Description of the *flip loci.*

Let σ_c be a critical value such that $\sigma_m < \sigma_c < \sigma_M$. Set

$$\sigma_c^+ = \sigma_c + \epsilon, \quad \sigma_c^- = \sigma_c - \epsilon,$$

where $\epsilon > 0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-, σ_c^+).

Lemma 2.4.1. *Let $\sigma_c \in [\sigma_m, \sigma_M]$ be a critical value. We define the flip loci $\mathcal{S}_{\sigma_c^\pm}$ as the set of triples in $\mathcal{N}_{\sigma_c^\pm}$ which are $\sigma_c^\pm$-stable but $\sigma_c^\mp$-unstable. Then*

$$\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}.$$

Proof. Let T be a triple representing a point in $\mathcal{N}_{\sigma_c}$. Let $T' \subset T$ be a proper subtriple then $\mathrm{par}\mu_{\sigma_c}(T') < \mathrm{par}\mu_{\sigma_c}(T)$. So for σ close to σ_c, $\mathrm{par}\mu_{\sigma}(T') < \mathrm{par}\mu_{\sigma}(T)$ by continuity. Therefore T is σ-stable for $\sigma = \sigma_c \pm \epsilon$, for $\epsilon > 0$ small. Hence, $\mathcal{N}_{\sigma_c} \subset \mathcal{N}_{\sigma_c^\pm} - \mathcal{S}_{\sigma_c^\pm}$.

Let T be a triple representing a point in $\mathcal{N}_{\sigma_c^\pm} - \mathcal{S}_{\sigma_c^\pm}$. Suppose $T \notin \mathcal{N}_{\sigma_c}$ then, there exists $T' \subset T$ such that $\mathrm{par}\mu_{\sigma_c}(T') \geq \mathrm{par}\mu_{\sigma_c}(T)$ but this is a contradiction with $\mathrm{par}\mu_{\sigma_c^\pm}(T') < \mathrm{par}\mu_{\sigma_c^\pm}(T)$ and $\mathrm{par}\mu_{\sigma_c}^{\mp}(T') \leq \mathrm{par}\mu_{\sigma_c}^{\mp}(T)$. □

Remark 2.4.2. The definition of $\mathcal{S}_{\sigma_c^\pm}$ can be extended for the maximal and minimal values of the parameter. Moreover, it is important to note that the lemma above does hold in those cases. Hence, the moduli space at σ_M is empty since $\mathcal{N}_{\sigma_M^+} = \emptyset$.

The following proposition can be found in [GGM].

Proposition 2.4.3. *Let $\sigma_c \in [\sigma_m, \sigma_M]$ be a critical value. Let $T = (E_1, E_2, \phi)$ be a triple which is σ_c-semistable. When $\sigma_c \in (\sigma_m, \sigma_M)$ one of the following is satisfied.*

(1) Suppose that T represents a point in $\mathcal{S}_{\sigma_c^+}$, i.e. suppose that T is σ_c^+-stable but σ_c^--unstable. Then T has a description as the middle term in an extension

$$0 \to T' \to T \to T'' \to 0 \tag{2.7}$$

in which

(a) T' and T'' are both σ_c^+-stable, with $\mathrm{par}\mu_{\sigma_c^+}(T') < \mathrm{par}\mu_{\sigma_c^+}(T)$,

(b) T' and T'' are both σ_c-semistable with $\mathrm{par}\mu_{\sigma_c}(T') = \mathrm{par}\mu_{\sigma_c}(T)$.

(2) *Similarly, if T represents a point in $\mathcal{S}_{\sigma_c^-}$, i.e. if T is σ_c^--stable but σ_c^+-unstable, then T has a description as the middle term in an extension (2.7) in which*

(a) *T' and T'' are both σ_c^--stable with $\operatorname{par}\mu_{\sigma_c^-}(T') < \operatorname{par}\mu_{\sigma_c^-}(T)$,*

(b) *T' and T'' are both σ_c-semistable with $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T)$.*

Lemma 2.4.4. *Let T' and T'' be triples which are σ-stable and of the same σ-slope, for some $\sigma \geq 2g-2$. Then*

$$\mathbb{H}^2(C^\bullet(T'', T')) = 0.$$

Proof. To prove that $\mathbb{H}^2(C^\bullet(T', T'')) = 0$, we have to check that the homomorphism

$$H^1(\mathrm{ParHom}\,(E_1'', E_1')) \oplus H^1(\mathrm{ParHom}\,(E_2'', E_2')) \to H^1(\mathrm{SParHom}\,(E_2'', E_1'(D)))$$

is onto. Equivalently, using Serre duality for parabolic bundles, we have to check that

$$H^0(\mathrm{ParHom}\,(E_1', E_2'')\otimes K) \xrightarrow{P} H^0((\mathrm{SParHom}\,(E_1', E_1'')\otimes K(D))\oplus(\mathrm{SParHom}\,(E_2', E_2'')\otimes K(D)))$$

is one-to-one. Let us suppose that P is not injective, therefore there is a map $\psi : E_1' \to E_2'' \otimes K$, $\psi \neq 0$, such that $P(\psi) = 0$. Denote $I = \mathrm{Im}\ \psi$ and $N = \ker(\psi)$. Hence $P(\psi) = 0$ implies that $I \subset \ker(\phi'' \otimes Id_K) = \ker(\phi'') \otimes K$ and also implies that $\mathrm{Im}\,(\phi') \subset N$. Give N and I the parabolic structures such that we can consider the following subtriples $(N, E_2', \phi') \subset (E_1', E_2', \phi')$ and $(0, I \otimes K^{-1}, 0) \subset (E_1'', E_2'', \phi'')$ of T' and T'' respectively.

Also we know that the sequence,

$$0 \to N \to E_1' \xrightarrow{\psi} I \to 0$$

is exact.

This together with the stability of T' and T'' gives the following inequalities

$$\frac{\mathrm{pardeg}\,(N) + \mathrm{pardeg}\,(E_2')}{\mathrm{rk}\,(N) + \mathrm{rk}\,(E_2')} + \sigma\frac{\mathrm{rk}\,(E_2')}{\mathrm{rk}\,(N) + \mathrm{rk}\,(E_2')} < \operatorname{par}\mu_\sigma(T')$$
$$\frac{\mathrm{pardeg}\,(E_1') - \mathrm{pardeg}\,(N)}{\mathrm{rk}\,(E_1') - \mathrm{rk}\,(N)} + (2-2g) < \operatorname{par}\mu_\sigma(T'').$$

Hence using that $\operatorname{par}\mu_\sigma(T') = \operatorname{par}\mu_\sigma(T'')$ and adding the two inequalities, we obtain

$$(\sigma + 2 - 2g)(\mathrm{rk}\,(E_1') - \mathrm{rk}\,(N)) < 0,$$

which implies that $\sigma < 2g-2$, contradicting our hypothesis. Therefore we get that P is injective for $\sigma \geq 2g-2$.

$\square$

Corollary 2.4.5. *$\mathcal{N}_\sigma$ is smooth and of the expected dimension, for any $\sigma \geq 2g-2$.* $\square$

Proposition 2.4.6. *If $\sigma_c > 2g-2$ then the loci $\mathcal{S}_{\sigma_c^\pm} \subset \mathcal{N}_{\sigma_c^\pm}$ have codimension greater than or equal to $-\chi(T', T'')$.*

Proof. Let us do the case of σ_c^+ (the other case is analogous). For simplicity we denote

$$\begin{aligned} \mathcal{N}'_{\sigma_c^\pm} &= \mathcal{N}_{\sigma_c^\pm}(r'_1, r'_2, d'_1, d'_2), \\ \mathcal{N}''_{\sigma_c^\pm} &= \mathcal{N}_{\sigma_c^\pm}(r''_1, r''_2, d''_1, d''_2). \end{aligned}$$

It follows from [Y2] that $\mathcal{N}'_{\sigma_c^\pm}$ and $\mathcal{N}''_{\sigma_c^\pm}$ are fine moduli spaces. That is, there are universal parabolic triples $\mathcal{T}' = (\mathcal{E}'_1, \mathcal{E}'_2, \Phi')$ and $\mathcal{T}'' = (\mathcal{E}''_1, \mathcal{E}''_2, \Phi)$ over $\mathcal{N}'_{\sigma_c^+} \times X$ and $\mathcal{N}''_{\sigma_c^+} \times X$ respectively. Thus we consider the complex $C^\bullet(\mathcal{T}'', \mathcal{T}')$ as defined in (2.2) and take relative hypercohomology with respect to the projection

$$\pi : X \times \mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+} \to \mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+}.$$

We define $W^+ := \mathbb{H}^1_\pi(C^\bullet(\mathcal{T}'', \mathcal{T}'))$. By Proposition 2.4.3, $S_{\sigma_c^+}$ is a subset of the locally trivial fibration $\mathbb{P}W^+$ over $\mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+}$.

The fibres of this fibration are projective spaces of dimension

$$\begin{aligned} \dim(\mathbb{P}(\operatorname{Ext}^1(T'', T'))) &= \dim \operatorname{Ext}^1(T'', T') - 1 \\ &= h^0(T'', T') + h^2(T'', T') - \chi(T'', T') - 1 \\ &= -\chi(T'', T') - 1. \end{aligned}$$

By Lemma 2.4.4 and Proposition 2.2.4, $h^0(T'', T') = 0 = h^2(T'', T')$. Therefore

$$\begin{aligned} \dim \mathcal{S}_{\sigma_c^+} &\leq -\chi(T'', T') + \dim(\mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+}) \\ &= -\chi(T'', T') - 1 + 1 - \chi(T', T') + 1 - \chi(T'', T'') \\ &= \dim \mathcal{N}_{\sigma_c^+} + \chi(T', T''), \end{aligned}$$

since the moduli spaces $\mathcal{N}'_{\sigma_c^+}$ and $\mathcal{N}''_{\sigma_c^+}$ are smooth of the expected dimension. We thus have $\dim \mathcal{N}_{\sigma_c^+} - \dim \mathcal{S}_{\sigma_c^+} \geq -\chi(T', T'')$. $\square$

If we prove that this codimension is positive then the number of irreducible components of $\mathcal{N}_\sigma$ does not vary when we cross a critical value bigger than $2g-2$.

2.5 Parabolic vortex equations.

Let us consider a smooth metric on X with Kähler form ω. Let $T = (E_1, E_2, \phi)$ be a parabolic triple. There is a condition for the existence of Hermitian metrics h_1 and h_2 on E_1 and E_2 respectively, adapted to their parabolic structures, satisfying

$$\sqrt{-1}\Lambda F(E_1) + \phi\phi^* = \tau_1 \operatorname{Id}_{E_1} \tag{2.8}$$

$$\sqrt{-1}\Lambda F(E_2) - \phi^*\phi = \tau_2 \operatorname{Id}_{E_2}, \tag{2.9}$$

where Λ is contraction with the Kähler form, ϕ^* is the adjoint of ϕ with respect to h_1 and h_2, Id_{E_1} and Id_{E_2} are the identity endomorphisms of E_1 and E_2 and τ_1 and τ_2 are real parameters. Since ϕ is singular at every point in D, these equations are only defined over $X - D$. The parameters τ_1 and τ_2 satisfy the following relation obtained by adding the traces of (2.8) and (2.9) and integrating,

$$\tau_1 r_1 + \tau_2 r_2 = \operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2). \tag{2.10}$$

We have the following.

Theorem 2.5.1 (Th.3.4 in [BG]). *Let $T = (E_1, E_2, \phi)$ be a parabolic triple. Let τ_1 and τ_2 satisfy $\tau_1 \operatorname{rk}(E_1) + \tau_2 \operatorname{rk}(E_2) = \operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2)$, and let $\sigma = \tau_1 - \tau_2$. Then E_1 and E_2 admit Hermitian metrics, adapted to the parabolic structures, satisfying (2.8) and (2.9) if and only if T is σ-polystable.* □

2.6 Codimension of the *flip loci*.

Let $\sigma_c \in (\sigma_m, \sigma_M]$ be a critical value and let T' and T'' be two σ_c-semistable parabolic triples which are σ_c^+-stable and such that $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T'')$. That is we are under the the conditions of Proposition 2.4.3 (1).

Suppose the parabolic bundles in T' and T'' are parabolic bundles of full flag type such that their weights are all different over each x i.e, $\alpha'^i_k(x) \neq \alpha''^j_l(x)$ for all $i,j = 1,2$, $k = 1,\ldots,r'_i$ and $l = 1,\ldots,r''_j$, so then $\operatorname{SParHom}(E'_i, E''_j) = \operatorname{ParHom}(E'_i, E''_j)$ for any i,j and $\operatorname{ParHom}(E'_i, E'_j) = \operatorname{SParHom}(E'_i, E'_j)$, $\operatorname{ParHom}(E''_i, E''_j) = \operatorname{SParHom}(E''_i, E''_j)$ when $i \neq j$. This is only a technical condition which does not reduce our general hypothesis since the extension T of T'' by T' has to be of full flag type on each of its parabolic bundles.

With the comment above, the study of extensions of triples is given by a complex such that

$$\begin{array}{rcl} C^\bullet(T'',T') : \mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2') & \xrightarrow{a_1} & \mathrm{ParHom}\,(E_2'',E_1'(D)) \\ (\xi_1,\xi_2) & \mapsto & \phi'\xi_2 - \xi_1\phi''. \end{array}$$

Let us denote $C_1 = \mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2')$ and $C_0(D) = \mathrm{ParHom}\,(E_2'',E_1'(D))$.

We want to find a bound for the Euler characteristic of the complex $C^\bullet(T'',T')$, that is,

$$\chi(C^\bullet(T'',T')) = (1-g)(\mathrm{rk}\,(C_1) - \mathrm{rk}\,(C_0)) + \deg(C_1) - \deg(C_0(D)).$$

In order to obtain bounds for $\deg(C_1)$ and $\deg(C_0)$, we exploit the properties of the triples T' and T'' involved in this complex.

The triples T' and T'' are $\sigma_c^\pm$-stable, so by Theorem 2.5.1 there are metrics such that

$$\begin{array}{ll} \sqrt{-1}\Lambda F(E_1') + \phi'(\phi')^* = \tau_1' \,\mathrm{Id}_{E_1'} & \sqrt{-1}\Lambda F(E_2') - (\phi')^*\phi' = \tau_2' \,\mathrm{Id}_{E_2'} \\ \sqrt{-1}\Lambda F(E_1'') + \phi''(\phi'')^* = \tau_1'' \,\mathrm{Id}_{E_1''} & \sqrt{-1}\Lambda F(E_2'') - (\phi'')^*\phi'' = \tau_2'' \,\mathrm{Id}_{E_2''} \end{array} \quad (2.11)$$

where $\sigma = \tau_1' - \tau_2' = \tau_1'' - \tau_2''$, and we can choose σ to be σ_c^+ or σ_c^- . In particular, $\tau_1' - \tau_1'' = \tau_2' - \tau_2''$. From Theorem 2.5.1 we know that the values for τ_1 and τ_2 are such that,

$$\begin{array}{rcl} \tau_1 & = & \mathrm{par}\mu_{\sigma}(T) \\ \tau_2 & = & \mathrm{par}\mu_{\sigma}(T) - \sigma. \end{array} \quad (2.12)$$

Let us consider the induced metrics on C_0 and C_1. Then

$$\begin{array}{rcl} F(C_0) & = & -F(E_2'')^t \otimes \mathrm{Id}_{E_1'} + \mathrm{Id}_{E_2''} \otimes F(E_1') \\ F(C_1) & = & \left(-F(E_1'')^t \otimes \mathrm{Id}_{E_1'} + \mathrm{Id}_{E_1''} \otimes F(E_1'), -F(E_2'')^t \otimes \mathrm{Id}_{E_2'} + \mathrm{Id}_{E_2''} \otimes F(E_2')\right). \end{array}$$

Here, we take the adapted metrics to the parabolic structures on each parabolic dual $(E_i')^*$, $(E_i'')^*$ and their parabolic tensors $E_i' \otimes^p E_i''$, induced by the adapted metrics on the bundles E_i' and E_i'', for $i = 1, 2$.

Let a_2 be defined by

$$\begin{array}{rcl} \mathrm{ParHom}\,(E_1'',E_2')(-D) & \xrightarrow{a_2} & \mathrm{ParHom}\,(E_1'',E_1') \oplus \mathrm{ParHom}\,(E_2'',E_2') \\ \xi & \to & (\phi'\xi, \xi\phi''). \end{array}$$

We define a_1^* as the adjoint of a_1 with respect to the hermitian adapted metric on C_0 induced by the hermitian adapted metrics on E_i', E_i'' for $i = 1, 2$. By computing the corresponding equations for C_0 and C_1, the connections on C_0 and C_1 satisfy the following,

$$\sqrt{-1}\Lambda F(C_0) + a_1 a_1^* = (\tau_1' - \tau_2'') \operatorname{Id}_{C_0} \tag{2.13}$$

$$\sqrt{-1}\Lambda F(C_1) - a_1^* a_1 + a_2 a_2^* = (\tau_1' - \tau_1'') \operatorname{Id}_{C_1}. \tag{2.14}$$

Lemma 2.6.1. *Let N and $Q(D)$ denote the kernel and cokernel of the homomorphism a_1. Then*

$$\operatorname{par}\mu(N) \le \operatorname{par}\mu_\sigma(T') - \operatorname{par}\mu_\sigma(T'') \tag{2.15}$$

$$\operatorname{par}\mu(Q) \ge \operatorname{par}\mu_\sigma(T'') - \operatorname{par}\mu_\sigma(T') + \sigma \tag{2.16}$$

Proof. The kernel N of a_1 is a subbundle of the hermitian bundle C_1, so that we may take the C^∞ orthogonal splitting $C_1 = N \oplus S$. Since N is a holomorphic subbundle, the induced connection D_N on N satisfies $D_{C_1}|_N = D_N + A$, where D_{C_1} is the connection on C_1 and $A \in \Omega^{1,0}(\operatorname{Hom}(N, S))$ is the second fundamental form of $N \subset C_1$. Therefore the curvature $F(N)$ of the connection on N satisfies $F(C_1)|_N = F(N) + \bar{A}^t \wedge A$.

We now use the equations (2.13) for the induced connection in N and take the trace and integrate on $X - D$ to get

$$\int_{X-D} \operatorname{Tr}(\sqrt{-1}\Lambda(F(N) + \bar{A}^t \wedge A) - a_1^* a_1|_N + a_2 a_2^*|_N) = \int_{X-D} \operatorname{Tr}((\tau_1' - \tau_1'') \operatorname{Id}_{C_1}|_N).$$

Thus giving

$$\operatorname{pardeg}(N) + \|A\|_{L^2}^2 + \int_{N-D} \operatorname{Tr}(a_2 a_2^*|_N) = (\tau_1' - \tau_1'') \operatorname{rk}(N).$$

and hence

$$\operatorname{pardeg}(N) \le (\tau_1' - \tau_1'') \operatorname{rk}(N). \tag{2.17}$$

Take the torsion-free part of $Q(D) \subset C_0(D)$, so that it is a bundle. This decreases the slope. Now, there is a C^∞ orthogonal splitting for C_0 such that $C_0 = S' \oplus Q$, where S' is the saturation of the image of a_1, which is holomorphic subbundle. The curvature of the induced connection on Q satisfies $F(C_0)|_Q = F(Q) + B \wedge \bar{B}^t$ with $B \in \Omega^{0,1}(\operatorname{Hom}(Q, S'))$. If we consider the equations for $F(C_0)$ restricted to Q, taking the trace and integrating, we obtain

$$\int_{X-D} \operatorname{Tr}(\sqrt{-1}\Lambda(F(Q) + B \wedge \bar{B}^t) + a_1 a_1^*)|_Q = \int_{X-D} \operatorname{Tr}((\tau_1' - \tau_2'') \operatorname{Id}_{C_0}|_Q),$$

and hence,

$$\operatorname{pardeg}(Q) - \|B\|^2_{L^2} = (\tau_1' - \tau_2'')(\operatorname{rk}(C_0) - \operatorname{rk}(\operatorname{Im}(a_1)).$$

Thus giving,

$$\operatorname{pardeg}(Q) \geq (\tau_1' - \tau_2'')(\operatorname{rk}(C_0) - \operatorname{rk}(\operatorname{Im}(a_1)). \tag{2.18}$$

Since $T = (E_1, E_2, \phi)$ are σ-stable triples (where $\sigma = \sigma_c^{\pm}$), we have $\sigma = \tau_1' - \tau_2' = \tau_1'' - \tau_2''$. Also, $\tau_1' = \operatorname{par}\mu_\sigma(T')$ and $\tau_1'' = \operatorname{par}\mu_\sigma(T'')$, by 2.12 and the bounds for N and Q follows from (2.17) and (2.18). □

Remark 2.6.2. The statement for Lemma 2.6.1 is also true for $\sigma = \sigma_c$, just passing the inequalities to the limit $\sigma \to \sigma_c$

Proposition 2.6.3. *Let X be a compact Riemann surface of genus $g > 0$, let D be the associate effective divisor to a finite set of marked points in X. Let T' and T'' be σ_c-semistable, σ_c^+-stable parabolic triples over X such that $\operatorname{par}\mu_\sigma(T') = \operatorname{par}\mu_\sigma(T'')$ for $\sigma \geq 2g-2$. Let $C^\bullet(T'', T')$ be the complex given in (2.2), if a_1 is not generically an isomorphism, then*

$$\chi(C^\bullet(T'', T')) < 1 - g.$$

Otherwise $\chi(C^\bullet(T'', T')) = 0$.

Proof.

$$\begin{aligned}
\chi(C^\bullet(T'', T')) &= (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + \deg(C_1) - \deg(C_0(D)) \\
&= (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + \deg(N) + \deg(\operatorname{Im}(a_1)) - \deg(C_0(D)) \\
&= (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + \deg(N) - \deg(Q) \\
&= (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + \deg(N) - \deg(Q(-D)(D)).
\end{aligned}$$

Observe that for any parabolic bundle E with set of points D, $\deg(E(D)) > \operatorname{pardeg}(E) \geq \deg(E)$, where the strict inequality is given by the fact that the weights on E always satisfy $\alpha_i(x) < 1$ for all i and for all $x \in D$. Using this, the hypothesis $\sigma \geq 2g - 2$, and Lemma 2.6.1, we have that

$$\begin{aligned}
\chi(C^\bullet(T'', T')) &< (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + \operatorname{pardeg}(K) - \operatorname{pardeg}(Q(-D)) \\
&= (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) - \sigma(\operatorname{rk}(C_0(D)) - \operatorname{rk}(\operatorname{Im}(a_1)) \\
&\leq (1-g)(\operatorname{rk}(C_1) - \operatorname{rk}(C_0)) + 2(1-g)(\operatorname{rk}(C_0) - \operatorname{rk}(\operatorname{Im}(a_1)) \\
&= (1-g)(\operatorname{rk}(C_1) + \operatorname{rk}(C_0) - 2\operatorname{rk}(\operatorname{Im}(a_1)).
\end{aligned}$$

If a_1 is not generically an isomorphism. Since $(\mathrm{rk}\,(C_1) + \mathrm{rk}\,(C_0) - 2\,\mathrm{rk}\,(\mathrm{Im}\,(a_1)) > 0$, we obtain

$$\chi(C^\bullet(T'', T')) < 1 - g. \tag{2.19}$$

If $\mathrm{rk}\,(C_0) = \mathrm{rk}\,(C_1)\,\mathrm{rk}\,(\mathrm{Im}\ a_1)$ and a_1 is not isomorphism, there is strict inequality in (2.19). Since $\deg(\mathrm{Im}\,(a_1)) < \deg(C_0(D))$. So $\chi(C^\bullet(T'', T')) < 0$. □

2.7 The case $\chi(C^\bullet(T'', T')) = 0$.

For $g > 0$ the codimension of the flip loci is less than zero when a_1 is not an isomorphism. We focus now our attention in the case when a_1 is an isomorphism.

Lemma 2.7.1. *If a_1 is generically an isomorphism of bundles, then either*

(a) $E_1'' = 0$ and $\phi' : E_2' \to E_1'$ is generically an isomorphism. In this case, $r_2 > r_1$.

(b) $E_2' = 0$ and $\phi'' : E_2'' \to E_1''$ is generically an isomorphism. In this case, $r_2 < r_1$.

Proof. One may look at a generic fibre where the maps ϕ' and ϕ'' are generic:

$$\begin{array}{ccc} \mathrm{ParHom}\,(\mathbb{C}^{r_1''}, \mathbb{C}^{r_1'}) \oplus \mathrm{ParHom}\,(\mathbb{C}^{r_2''}, \mathbb{C}^{r_2'}) & \to & \mathrm{ParHom}\,(\mathbb{C}^{r_2''}, \mathbb{C}^{r_1'}) \\ (a, b) & \mapsto & \phi'\xi_2 - \xi_1\phi'' \end{array}$$

Suppose ϕ'' is not surjective, we can take $\xi_1 \neq 0$ with $\xi_1|_{\mathrm{Im}\,(\phi'')} = 0$ and $\xi_2 = 0$. They are such that $a_1(a, b) = 0$ and this contradicts to a_1 being a generic isomorphism.

If we suppose that ϕ' is not injective, we can take $\xi_2 \neq 0$ with $\mathrm{Im}\,(\xi_2) \subset \ker\phi'$ and $\xi_1 = 0$, to get $a_1(a, b) = 0$.

Now take a map $\mathbb{C}^{r_2''} \to \mathbb{C}^{r_1'}$ which induces a non-zero map $\ker(\phi'') \to \mathrm{coker}\,(\phi')$. This cannot be in the image of a_1. So either ϕ' or ϕ'' are isomorphisms. In the first case $r_1' r_1'' + r_2' r_2'' = r_2'' r_1'$ gives $r_1'' = 0$ and we are in case (a). In the second, we are in case (b).

□

From the lemma above we get that if a_1 is a generic isomorphism then $r_1 \neq r_2$. The following proposition will be used to restrict our study to $r_1 > r_2$.

Throughout this section we assume that $r_1 > r_2$. The case $r_1 < r_2$ reduces to the previous one by duality.

Proposition 2.7.2. *If the moduli space of σ-stable triples with $r_1 \neq r_2$ is non empty for $\sigma \geq 2g-2$, then there is a critical value $\sigma_L \in (2g-2, \sigma_M]$. Moreover the triples in $\mathcal{S}_{\sigma_L^-}$ are extensions of triples T' and T'' satisfying $\chi(C^\bullet(T'', T')) = 0$.*

Proof. By Lemma 2.4.1 and Proposition 2.3.2, the moduli spaces of σ stable triples are isomorphic unless they cross a critical value of the parameter σ. Hence, as $\mathcal{N}_\sigma$ is empty for $\sigma > \sigma_M$ and there is some σ such that the moduli space is non empty, then there exist a critical value (at least one) σ_L such that the triples in $\mathcal{S}_{\sigma_L^-}$ are extensions of triples T' and T'' satisfying $\chi(C^\bullet(T'', T')) = 0$. □

We focus our work on describing the triples in $\mathcal{S}_{\sigma_L^-}$.

Let $T = (E_1, E_2, \phi)$ be a triple on the flip locus $S_{\sigma_c^-}$, by Proposition 2.4.3 it has a description as the middle term of an extension

$$0 \to T' \to T \to T'' \to 0,$$

such that T' and T'' are both σ_c^--stable with $\operatorname{par}\mu_{\sigma_c^-}(T') < \operatorname{par}\mu_{\sigma_c^-}(T)$ and T' and T'' are σ_c-semistable with $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T)$.

Remark 2.7.3. The triples we are interested in are identified with parabolic Higgs bundles at $\sigma = 2g-2$. Hence, as for the study of connected components we are restricting our study to parabolic bundles V and W with different weights, we assume from now on that the triples we consider have different weights on each fibre $E_{1,x} \oplus E_{2,x}$. This will allow us to identify $\operatorname{ParHom}(E_i, E_j) = \operatorname{SParHom}(E_i, E_j)$ for $i \neq j$ and $i, j = 1, 2$.

Proposition 2.7.4. *Assume $r_1 > r_2$, and let T', T'' be σ-semistable, σ^{pm}-stable triples with $\mu_\sigma(T') = \mu_\sigma(T'')$, for some $\sigma \geq 2g-2$. Write for $x \in D$, $\phi'' = z^{-1}(\phi_0 + \phi_1 z + \phi_2 z^2 + \cdots)$ where z is a local holomorphic coordinate around p in X. Then $\chi(C^\bullet(T'', T')) = 0$ (equivalently a_1 is an isomorphism) if and only if for any extension $T' \to T \to T''$ the following holds,*

1. *$E_2' = 0$. So $E_2 = E_2''$.*
2. *$\phi'' : E_2 \to E_1''(D)$ is a fibre bundle isomorphism at $X - D$.*
3. *For all $x \in D$, the map $\operatorname{ParHom}(E''_{1,x}, E'_{1,x}) \to \operatorname{ParHom}(E_{2,x}, E'_{1,x})$, $f \mapsto -f \circ \phi_0$, is surjective.*
4. *For all $x \in D$, the induced homomorphism $\phi_1 : \ker \phi_0 \to \operatorname{coker} \phi_0$ gives us a map $\operatorname{ParHom}(\operatorname{coker} \phi_0, E'_{1,x}) \to \operatorname{Hom}(\ker \phi_0, E'_{1,x})$, $f \mapsto -f \circ \phi_1$, that is surjective.*

Proof. By Proposition 2.6.3, $\chi(C^\bullet(T'',T'))=0$ if and only if a_1 is an isomorphism. And, by Lemma 2.7.1 together with the assumption that $r_1>r_2$ we get $E'_2=0$. This proves (1).

Lemma 2.7.1 also says that $\phi'': E''_2 \to E''_1(D)$ is generically an isomorphism, this gives that $r''_2=r''_1$ and using that

$$0=\chi(C^\bullet(T'',T'))=\deg(\mathrm{ParHom}\,(E''_1,E'_1))-\deg(\mathrm{ParHom}\,(E_2,E'_1(D)))$$

we get that the two bundles involved in the complex $C^\bullet(T'',T')$ must be of the same rank and of the same degree. Hence,

$$\mathrm{ParHom}\,(E''_1,E'_1)\xrightarrow{a_1}\mathrm{ParHom}\,(E_2,E'_1(D))$$

where $a_1(\xi)=-\xi\circ\phi''$, is an isomorphism of bundles.

We want to prove (2) i.e., we want to prove that $\phi'': E''_2\to E''_1(D)$ is an isomorphism of bundles out of D. So, we take $U\subset X$ an open set such that there is no point of D in it. As we just said above a_1 is an isomorphism of bundles then we have that $\mathrm{Hom}\,(E''_1,E'_1)|_U\to \mathrm{Hom}\,(E_2,E'_1(D))|_U$ is an isomorphism over each U. So $\phi''|_U: E_2|_U\to E''_1(D)|_U$ is an isomorphism. This proves (2).

Now, let $x\in D$. In a neighbourhood U of x take a coordinate z vanishing at x and write ϕ'' over U as $\phi''=\phi_0 z^{-1}+\phi_1+\phi_2 z+\cdots$, where $\phi_0\in\mathrm{ParHom}\,(E_{2,x},E''_{1,x})$ and $\phi_i\in\mathrm{Hom}\,(E_{2,x},E''_{1,x})$ for $i=1,2,\ldots$ As a_1 is an isomorphism of bundles, then

$$a_1|_x:\mathrm{ParHom}\,(E''_1,E'_1)_x\to\mathrm{ParHom}\,(E_2,E'_1(D))_x$$

is a isomorphism of vector spaces.

Consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccc}
\dfrac{\mathrm{Hom}\,(E''_{1,x},E'_{1,x})}{\mathrm{ParHom}\,(E''_{1,x},E'_{1,x})}\otimes\mathcal{O}(-x) & \to & \mathrm{ParHom}\,(E''_1,E'_1)_x & \to & \mathrm{ParHom}\,(E''_{1,x},E'_{1,x}) \\
\downarrow b_0 & & \downarrow b_1 & & \downarrow b_2 \\
\dfrac{\mathrm{Hom}\,(E_{2,x},E'_{1,x})}{\mathrm{ParHom}\,(E_{2,x},E'_{1,x})} & \to & \mathrm{ParHom}\,(E_2,E'_1(x))_x & \to & \mathrm{ParHom}\,(E_{2,x},E'_{1,x})\otimes\mathcal{O}(x).
\end{array}$$

where b_1 is induced by a_1 that is, $f\to -f\circ\phi''$. This implies that b_2 is $f_0\to-(f_0\circ\phi_0)z^{-1}$ and b_0 is $f_1\to-(f_1\circ\phi_0)z^{-1}$. The long exact sequence produced by the snake lemma,

$$0\to\ker(b_0)\to\ker(b_1)\to\ker(b_2)\xrightarrow{\delta}\mathrm{coker}\,(b_0)\to\mathrm{coker}\,(b_1)\to\mathrm{coker}\,(b_2)\to 0,$$

shows us that b_1 being surjective is equivalent to b_2 being surjective and the connecting homomorphism , $\delta:\ker(b_2)\to\mathrm{coker}\,(b_0)$, also being surjective. The condition that b_2 is surjective is (3).

For the remaining condition, we need to describe explicitly the connecting homomorphism δ. Take $f_0 \in \mathrm{ParHom}\,(E''_{1,x}, E'_{1,x})$ lying in

$$\ker b_2 = \mathrm{ParHom}\,(E''_{1,x}/\phi_0(E_{2,x}), E'_{1,x}).$$

Lift f_0 to a local section of $\mathrm{ParHom}\,(E''_1, E'_1)$ on U, e.g. taking $f(z) \equiv f_0$. Compose with ϕ'' to get $-(f\circ\phi_0 + f\circ\phi_1 z + \cdots)z^{-1}$. Recalling that $f\circ\phi_0 = 0$, the leading term is

$$-f_0\circ\phi_1 \in \mathrm{coker}\ b_0 = \frac{\mathrm{Hom}\,(E_{2,x}, E'_{1,x})}{\mathrm{ParHom}\,(E_{2,x}, E'_{1,x}) + b_0(\mathrm{Hom}\,(E''_{1,x}, E'_{1,x}))}.$$

Considering that (3) is true we get that $\mathrm{ParHom}\,(E_{2,x}, E'_{1,x}) \subset b_0(\mathrm{ParHom}\,(E''_{1,x}, E'_{1,x}))$, since the maps b_0 and b_2 are both composition with ϕ_0. Hence, the image of f_0 under the connecting homomorphism is $\mathrm{Hom}\,(E''_{1,x}, E'_{1,x})$, so

$$-f_0\circ\phi_1 \in \mathrm{coker}\ b_0 = \frac{\mathrm{Hom}\,(E_{2,x}, E'_{1,x})}{\mathrm{Hom}\,(E''_{1,x}, E'_{1,x})} = \mathrm{Hom}\,(\ker\phi_0, E'_{1,p})$$

Therefore, the condition for the connecting homomorphism δ of being surjective is equivalent to item (4) and, we have proved above that δ is surjective since b_1 is surjective. □

The following lemmas will show us that conditions in Proposition 2.7.4 give us explicitly the topological type of the triples T' and T'' satisfying $\chi(C^\bullet(T'', T')) = 0$.

Lemma 2.7.5. *Let $\phi_0 \in \mathrm{ParHom}\,(E_{2,x}, E''_{1,x})$ be such that $\phi'' = z^{-1}(\phi_0 + \phi_1 z + \cdots)$, then the condition (4) in Proposition 2.7.4 is equivalent to having all the weights of $E'_{1,x}$ bigger than those of* $\mathrm{coker}\,(\phi_0)$*, and $\phi_1 : \ker\phi_0 \to \mathrm{coker}\ \phi_0$ being an isomorphism.*

Proof. The condition (4) of Proposition 2.7.4 says that

$$\mathrm{ParHom}\,(\mathrm{coker}\,(\phi_0), E'_{1,x}) \to \mathrm{Hom}\,(\ker(\phi_0), E'_{1,x})$$

defined by $f \to -f\circ\phi_1$, where $\phi_1 \in \mathrm{Hom}\,(E_2, E'_1)$, is surjective. Since $\mathrm{rk}\,(E''_1) = \mathrm{rk}\,(E_2)$ $\ker(\phi_0)$ and $\mathrm{coker}\,(\phi_0)$ are vector spaces of the same dimension. This and condition (4) implies that $\phi_1 : E_{2,x} \to E''_{1,x}$ is such that $\phi_1|_{\ker(\phi_0)} : \ker(\phi_0) \to \mathrm{coker}\,(\phi_0)$ is an isomorphism. Hence, as $\mathrm{Hom}\,(\ker(\phi_0), E'_{1,x}) \cong \mathrm{Hom}\,(\mathrm{coker}\,(\phi_0), E'_{1,x})$, we get that $\mathrm{ParHom}\,(\mathrm{coker}\,(\phi_0), E'_{1,x})$ is equal to $\mathrm{Hom}\,(\mathrm{coker}\,(\phi_0), E'_{1,x})$, which give us the statement that all the weights of $\mathrm{coker}\,(\phi_0)$ are smaller than the weights of $E'_{1,x}$. □

Lemma 2.7.6. *Suppose that $\phi_0 : E_{2,x} \to E_{1,x}$ is a generic parabolic homomorphism, and suppose that $E_{1,x} = E'_{1,x} \oplus E''_{1,x}$ and $\mathrm{Im}\ \phi_0 \subset E''_{1,x}$. Then (3) is satisfied.*

Proof. Suppose that ϕ_0 is generic as an element in $\mathrm{ParHom}\,(E_{2,x}, E_{1,x})$, and let us see tha the map $\mathrm{ParHom}\,(E''_{1,x}, E'_{1,x}) \to \mathrm{ParHom}\,(E_{2,x}, E'_{1,x})$, $f \mapsto -f \circ \phi_0$, is surjective. Take $g \in \mathrm{ParHom}\,(E_{2,x}, E'_{1,x})$. Consider the map $\phi_\epsilon = \phi_0 \oplus \epsilon g : E_{2,x} \to E''_{1,x} \oplus E'_{1,x}$. For ϵ small we have that ϕ_ϵ also lives in the generic open set, so it is equivalent to ϕ_0 by the action of $\mathrm{ParAut}\,(E_{2,x}) \times \mathrm{ParAut}\,(E_{1,x})$. This means that

$$\begin{pmatrix} a_\epsilon & b_\epsilon \\ c_\epsilon & d_\epsilon \end{pmatrix} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} M_\epsilon = \begin{pmatrix} \phi_0 \\ \epsilon g \end{pmatrix}.$$

Both matrices are the identity for $\epsilon = 0$, so a_ϵ is invertible for small ϵ. Therefore $\phi_0 M_\epsilon = a_\epsilon^{-1}\phi_0$ and $c_\epsilon \phi_0 M_\epsilon = \epsilon g$ so

$$g = \epsilon^{-1} c_\epsilon a_\epsilon^{-1} \phi_0$$

as required. □

Lemma 2.7.7. *Assume $r_1 = n$, $n > 1$, and $r_2 = 1$. For a fixed topological type of T there is at most one distribution of weights and choices of degrees, i.e. topological types, for T' and T'' such that $\chi(T'', T') = 0$.*

Proof. Let T be a triple in $\mathcal{S}_{\sigma_c^\pm}$. Then it is an extension of T' and T'' such that $\mathrm{par}\mu_{\sigma_c}(T') = \mathrm{par}\mu_{\sigma_c}(T) = \mathrm{par}\mu_{\sigma_c}(T'')$.

We must choose T' and T'' of the following form:

$$\begin{array}{ccccccccc} & & & & E_2 & \xrightarrow{=} & E''_2 & & \\ & & & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & E'_1(D) & \longrightarrow & E_1(D) & \xrightarrow{\pi} & E''_1(D) & \longrightarrow & 0 \end{array}$$

That is, given $T = (E_1, E_2, \phi)$ we choose T'' such that $E''_2 = E_2$, E''_1 is a vector bundle of rank $r_2 = 1$, and degree $\deg(E''_1) = \deg(E_2) + \deg(\mathrm{coker}\,(\phi''))$. As system of weights, E''_1 has some of the weights of E_1 (we will see later how to choose them) . For T' choose $E'_2 = 0$ and E'_1 is a parabolic bundle of rank $r_1 - r_2$ and degree $\deg(E_1) - \deg(E_2) - \deg(\mathrm{coker}\,(\phi''))$, its system of weights will be described now.

At each $x \in D$, we write $\phi'' = z^{-1}(\phi_0 + \phi_1 z + \cdots)$. Assume $r_1 = n$ and $r_2 = 1$. At each $x \in D$, $E_{1,x} = E'_{1x} \oplus E''_{1,x}$, where $E_{1,x} = \mathrm{Im}\ \phi_0 \oplus \mathrm{coker}\ \phi_0$. In order to satisfy condition (3) we need that

$$\mathrm{ParHom}\,(E''_{1,x}, E'_{1,x}) = \mathrm{ParHom}\,(\mathbb{C}_{\alpha_i^1(x)}, \mathbb{C}^{n-1}_{\alpha_1^1(x),\ldots,\hat{\alpha}_i^1(x),\ldots,\alpha_n^1})$$
$$\mathrm{ParHom}\,(E_{2,x}, E'_{1,x}) = \mathrm{ParHom}\,(\mathbb{C}_{\alpha^2(x)}, \mathbb{C}^{n-1}_{\alpha_1^1(x),\ldots,\hat{\alpha}_i^1(x),\ldots,\alpha_n^1})$$

where $\alpha_i^1(x)$ is the weight of $E''_{1,x}$, $\alpha^2(x)$ the weight of $E_{2,x}$ and the other $\alpha_i^1(x)$ the weights of $E'_{1,x}$, with $\dim \mathrm{ParHom}\,(E''_{1,x}, E'_{1,x}) \geq \dim \mathrm{ParHom}\,(E_{2,x}, E'_{1,x})$. This can happen if $\alpha^2(x) > \alpha_n^1(x)$ in which case $\phi_0 = 0$. If $\alpha^2(x) < \alpha_n^1(x)$ then it must be $\alpha_{i-1}^1(x) < \alpha^2(x) < \alpha_{i+1}^1(x)$ since $\dim(\mathrm{ParHom}\,((E_{2,x}, E'_{1,x}) \leq \dim \mathrm{ParHom}\,(E''_{1,x}, E'_{1,x})$. Now if $\alpha^2(x) > \alpha_i^1(x)$ then $\phi_0 = 0$, $\phi_0 \in \mathrm{ParHom}\,(E_{2,x}, E''_{1,x})$, so (3) is not satisfied, therefore $\alpha_{i-1}(x) < \alpha^2(x) < \alpha_i(x)$.

Condition (4) tell us that the weights in E'_1 have to be bigger than the weights in $\mathrm{coker}\,(\phi_0)$. We have three cases of how ϕ_0 must be, this are:

$$\begin{pmatrix} 0 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ 0 & & & \end{pmatrix} \begin{pmatrix} * & & & \\ * & & & \\ * & & & \\ * & & & \\ * & & & \end{pmatrix} \begin{pmatrix} 0 & & & \\ 0 & & & \\ \vdots & & & \\ * & & & \\ * & & & \end{pmatrix}$$

In the first case, $\phi_0 = 0$ and it implies that $\alpha^2(x) > \alpha_n^1(x)$, by discussion above. Hence, the weight $\alpha_i^1(x)$ of $E''_{1,x}$ is the weight corresponding to the cokernel of ϕ_0 and must be less than the weights of $E'_{1,x}$. This forces $i = 1$.

In the second case, $\mathrm{coker}\,(\phi_0) = 0$ and $\alpha^2(x) < \alpha_1^1(x)$ hence, condition (4) is satisfied automatically and $i = 1$.

In the third case, $\mathrm{Im}\,(\phi_0)$ has weight $\alpha_i^1(x)$ and $\alpha_{i-1}^1(x) < \alpha^2(x) < \alpha_i^1(x)$. So i is the minimum k such that $\alpha^2(x) < \alpha_k^1(x)$.

Under these choices Lemma 2.7.6 and Lemma 2.7.5 are satisfied by the triples T' and T'' constructed above, hence these triples satisfy $\chi(T'', T') = 0$ by Proposition 2.7.4. □

It is possible to extend the above argument to the case $r_2 > 1$. This is left for future work.

Remark 2.7.8. Note that the case $n = 1$ is much simpler and can be done directly with a similar reasoning as the one in the proof of Proposition 2.8.3

Proposition 2.7.9. *The maximal critical value of σ for the family of moduli spaces with topological type (r_1, r_2, d_1, d_2), where $r_1 > r_2$ and, weight type (α^1, α^2) is*

$$\sigma_L = \left(1 + \frac{r_1 + r_2}{r_1 - r_2}\right) (\mathrm{par}\mu\,(E_1) - \mathrm{par}\mu\,(E_2)) + \frac{A}{r_2} \frac{r_1 + r_2}{(r_1 - r_2)} \tag{2.20}$$

where A is, the difference from the sum of the weights on E_1 minus the sum of the weights on $\mathrm{Im}\,(\phi_0)$, and it is bounded above by $r_2 \cdot s$.

Proof. Take T' and T'' in $\mathcal{S}_{\sigma_L}$. They satisfy $\chi(T'', T') = 0$. Also, by Proposition 2.4.3 T' satisfies $\mathrm{par}\mu\,(T')_{\sigma_L} = \mathrm{par}\mu_{\,\sigma_L}(T)$. We get the value of σ_L using this equality. $\square$

Remark 2.7.10. Note that

$$\sigma_L < \sigma_M = \left(1 + \frac{r_1 + r_2}{r_1 - r_2}\right) (\mathrm{par}\mu\,(E_1) - \mathrm{par}\mu\,(E_2)) + s\frac{r_1 + r_2}{r_1 - r_2},$$

as required by Proposition 2.1.9.

2.8 Birationality and irreducibility of moduli spaces.

Theorem 2.8.1. *Let $g \geq 1$ be the genus of X. Let $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ be the moduli space of σ-stable parabolic triples with full flags on its parabolic structures. Then, for $\sigma \in [2g-2, \sigma_L)$ the moduli spaces $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ are birational. In particular, they have the same number of irreducible components.*

Proof. From Proposition 2.4.6 and Proposition 2.6.3, the flip loci at a critical value $\sigma \in [2g-2, \sigma_L)$ is positive if and only if $g > 0$ and a_1 is not generically an isomorphism. Otherwise C_0 and $C_1(D)$ are isomorphic as bundles over $X - D$ hence $\chi(C^\bullet(T'', T')) = 0$, but this only happens for σ_L. The result follows. $\square$

Since we have proved that birationality holds for the moduli spaces $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ for $\sigma \in [2g-2, \sigma_L)$, it only remains to prove that the moduli space $\mathcal{N}_{\sigma_L}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ at the maximal critical value of σ is irreducible.

Proposition 2.8.2. *The moduli space of σ_L-stable parabolic triples of type $(n, 1, a, b, \alpha^1, \alpha^2)$ is a locally trivial fibration over the space of parabolic bundles times the moduli of parabolic triples of type $(1, 1, b, b, \alpha^2, \alpha')$ where α' is determined by σ_L.*

Proof. From the arguments in the proof of Proposition 2.4.6 we know that the flip loci $\mathcal{S}_{\sigma_c^-}$ is an open subset of the locally trivial fibration $\mathbb{P}W^-$ over $\mathcal{N}'_{\sigma_c^-} \times \mathcal{N}''_{\sigma_c^-}$ where $W^- := \mathbb{H}^1_\pi(C^\bullet(T'', T')$ as in the proof of the Proposition 2.4.6.

From Lemma 2.7.7 we obtain for $\sigma_c = \sigma_L$ that the moduli spaces $\mathcal{N}'_{\sigma_L^-}$ and $\mathcal{N}''_{\sigma_c^-}$ are the moduli space of parabolic bundles and the moduli space of σ_L^--stable triples $\mathcal{N}_{\sigma_L}(1, 1, b, b, \alpha^2, \alpha')$ where the system of weights α' is given explicitly in Lemma 2.7.7. $\square$

Proposition 2.8.3. *Let $\mathcal{N}_{\sigma_L^-}(1,1,d_2,d_2,\alpha^2,\alpha')$ be the moduli space of σ_L-stable parabolic triples. Then the following map is an isomorphism*

$$\begin{array}{rcl} \mathcal{N}_{\sigma_L^-}(1,1,d_2,d_2,\alpha^2,\alpha') & \to & \operatorname{Jac}^{d_2} X \times S^m X \\ (E_1,E_2,\phi) & \mapsto & (E_2, \operatorname{div}(\phi)) \end{array}$$

where $m = \deg(\operatorname{SParHom}(E_2, E_1(D))$. □

Proof. Assume that $\mathcal{N}_{\sigma_L^-}(1,1,d_2,d_2,\alpha^2,\alpha')$ is non empty and denote $S = \{x \in D | \ \alpha^2(x) > \alpha'(x)\}$. For any line bundle E_2 in $\operatorname{Jac}^{d_2} X$ and effective divisor $D' \in S^m X$ we get a line bundle $M = \mathcal{O}(D')$ with a non-zero section ϕ determined up to multiplication by nonzero scalars. We then obtain a parabolic triple by letting

$$E_1 = E_2 \otimes \mathcal{O}(S - D) \otimes M.$$

It follows from this construction that the homomorphism $\mathcal{N}_{\sigma_L^-}(1,1,d_2,d_2,\alpha^2,\alpha') \to \operatorname{Jac}^{d_2} X \times S^m X$ is surjective. To see that is it also injective, we note that taking non-zero scalar multiples of the section ϕ give rise to isomorphic parabolic triples. Thus the map give is an isomorphism.

□

Finally, the Proposition 2.8.3 and the Proposition 2.8.2 yields the following.

Theorem 2.8.4. *Let X be a compact Riemann surface with genus $g \geq 1$. Let D be a effective divisor associated to a finite set of marked points on X. Then the moduli space $\mathcal{N}_{\sigma_L^-}(n,1,d_1,d_2;\alpha^1,\alpha^2)$ of σ_L^--stable parabolic triples with generic weights and full flags on its parabolic structures, is a non empty and irreducible algebraic variety.*

2.9 Triples of equal rank

Proposition 2.9.1. *Assume $r_1 = r_2$. There is a value σ_1 such that any σ-stable parabolic triple with $\sigma > \sigma_1$ has ϕ injective, hence*

$$0 \to E_2 \to E_1(D) \to \mathcal{T} \to 0,$$

where $\mathcal{T}$ is a torsion sheaf.

Proof. Denote $N = \ker\phi$ and consider the parabolic subtriple $(0, N, \phi)$. Suppose that $k = \mathrm{rk}\,(N) > 0$. The σ-stability of T implies that

$$\mathrm{pardeg}\ N + k\sigma < k\left(\frac{\mathrm{pardeg}\,(E_1 \oplus E_2)}{2r_1} + \frac{1}{2}\sigma\right).$$

Now consider the subtriple (I, E_2, ϕ) where $I(D)$ is the parabolic image sheaf of ϕ of rank $\mathrm{rk}\,(I) = r_1 - k$. The σ-stability of T gives us

$$\mathrm{pardeg}\,(I \oplus E_2) + r_1\sigma < (2r_1 - k)\left(\frac{\mathrm{pardeg}\,(E_1 \oplus E_2)}{2r_1} + \frac{1}{2}\sigma\right).$$

Adding up both equations, and noting that $\mathrm{pardeg}\ N + \mathrm{pardeg}\ I(D) = \mathrm{pardeg}\ E_2$, we get

$$2\,\mathrm{pardeg}\ E_2 - (r_1 - k)s + (r_1 + k)\sigma < \mathrm{pardeg}\,(E_1 \oplus E_2) + r_1\sigma,$$

which is rewritten as

$$\sigma \leq \frac{1}{k}\left(\mathrm{pardeg}\ E_1 - \mathrm{pardeg}\ E_2 + (r_1 - k)s\right).$$

So for $\sigma_1 = \mathrm{pardeg}\ E_1 - \mathrm{pardeg}\ E_2 + (r_1 - 1)s$ the result follows. □

Lemma 2.9.2. *Assume $r_1 = r_2$ (denote it by r) and $\sigma > \sigma_1$. Suppose that T is σ-stable and T' is a subtriple of T with $r_1' = r_2'$. Write $E_2 \to E_1(D) \to \mathrm{coker}\,(\phi)$, $E_2' \to E_1'(D) \to \mathrm{coker}\,(\phi')$, $t = \deg(\mathrm{coker}\,(\phi))$, $t' = \deg(\mathrm{coker}\,(\phi'))$. Then*

$$\begin{aligned} \mathrm{par}\mu\,(E_1') &< \mathrm{par}\mu\,(E_1) + \frac{1}{2}\left(\frac{t'}{r'} - \frac{t}{r}\right) \\ \mathrm{par}\mu\,(E_2') &< \mathrm{par}\mu\,(E_2) - \frac{1}{2}\left(\frac{t'}{r'} - \frac{t}{r}\right) \end{aligned}$$

Proof. From Proposition 2.9.1, as $\sigma > \sigma_1$, the triple is such that ϕ is injective, so ϕ' is injective for all subtriple T' of T. Hence for a subtriple T' with $r_1' = r_2' = r'$ we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_2' & \longrightarrow & E_1'(D) & \longrightarrow & \mathrm{coker}\,(\phi') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_2 & \longrightarrow & E_1(D) & \longrightarrow & \mathrm{coker}\,(\phi) & \longrightarrow & 0, \end{array}$$

where coker (ϕ) and coker (ϕ') are torsion sheaves with lengths t and t' respectively. By stability,

$$\begin{aligned}
0 &> \operatorname{par}\mu_\sigma(T') - \operatorname{par}\mu_\sigma(T) \\
&= \frac{1}{2}(\operatorname{par}\mu(E_1') + \operatorname{par}\mu(E_2') - \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2)) \\
&= \operatorname{par}\mu(E_1') - \operatorname{par}\mu(E_1) - \frac{1}{2}(\operatorname{par}\mu(E_1') - \operatorname{par}\mu(E_2')) + \frac{1}{2}(\operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2)) \\
&= \operatorname{par}\mu(E_2') - \operatorname{par}\mu(E_2) + \frac{1}{2}(\operatorname{par}\mu(E_1') - \operatorname{par}\mu(E_2')) - \frac{1}{2}(\operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2)).
\end{aligned}$$

Now at each point $x \in D$, $|\sum \alpha_j^1 - \sum \alpha_i^2| \le r$, so $t \le$ pardeg $E_1(D) -$ pardeg $E_2 \le t + r_1 s$, equivalently $t - rs \le$ pardeg $E_1 -$ pardeg $E_2 \le t$ or $t/r - s \le \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2) \le t/r$. Substituting into the formula above, we get the result in the statement. □

Proposition 2.9.3. *Assume $r_1 = r_2$. There is a value σ_2 such that $\mathcal{N}_\sigma = \mathcal{N}_{\sigma'}$ for any $\sigma, \sigma' \ge \sigma_2$ (i.e. there are no critical values above σ_2).*

Proof. Consider $T \in \mathcal{N}_\sigma(d_1, d_2, r, r; \alpha^1, \alpha^2)$ with $\sigma > \sigma_1$. Suppose that T is properly σ_c-semistable and let $T' \subset T$ be the destabilising triple. Clearly $r_2' \le r_1'$, since the triple T is injective, hence also is T'. On the other hand, they are not equal, since then T would be σ-semistable for generic values of σ and could not be σ-stable for some σ. In the formula

$$\sigma_c = 2\operatorname{par}\mu(E_1')\frac{r_1'}{r_1' - r_2'} + 2\operatorname{par}\mu(E_2')\frac{r_2'}{r_1' - r_2'} - (\operatorname{par}\mu(E_1) + \operatorname{par}\mu(E_2))\frac{r_1' + r_2'}{r_1' - r_2'} \tag{2.21}$$

we want to bound the values of $\operatorname{par}\mu(E_1')$ and $\operatorname{par}\mu(E_2')$ in order to get a bound for the critical value σ_c independent of T.

Apply Lemma 2.9.2 to the subtriples $(\phi'(E_2')(-D), E_2', \phi')$ and $(E_1', (\phi')^{-1}(E_1'(D)), \phi')$, both of which satisfy the equal rank condition. The first one has no torsion, the second has torsion with $0 \le t' \le t$. We get

$$\begin{aligned}
\operatorname{par}\mu(E_2') &< \operatorname{par}\mu(E_2) + \frac{t}{2r} \\
\operatorname{par}\mu(E_1') &< \operatorname{par}\mu(E_1) + \frac{1}{2}\left(\frac{t'}{r'} - \frac{t}{r}\right) + s \le \operatorname{par}\mu(E_1) + \frac{t(r - r')}{2rr'}.
\end{aligned}$$

Using that $\frac{t}{r} \le \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2) + s$ and $1 \le r' \le r - 1$ we get bounds on $\operatorname{par}\mu(E_1')$ and $\operatorname{par}\mu(E_2')$. Substituting these bounds into (2.21) and using that $r_1 - r_2 \ge 1$ and $r_1, r_2 \le r$, we get a bound on σ_c, as required. □

Corollary 2.9.4. *For $r_1 = r_2$, all moduli spaces $\mathcal{N}_\sigma$, for $\sigma \geq 2g-2$ are birational to each other.*

Proof. This is a consequence of Theorem 2.4.6. For $\chi(T'', T') = 0$ it must be a_1 an isomorphism. But this is impossible if $r_1 = r_2$ by Lemma 2.7.1 $\square$

Let $\mathcal{N}_L(r, r, d_1, d_2; \alpha^1, \alpha^2)$ be the moduli space of σ_L-stable parabolic triples with fixed topological type and fixed system of weights, where σ_L means a large value of the parameter σ. That is, $\sigma_L > \sigma_2$.

Lemma 2.9.5. *$\mathcal{N}_L(r, r, d_1, d_2; \alpha^1, \alpha^2) \neq \emptyset$ implies that $d_1 + rs - d_2 \geq \sum_{x \in D} r_x$.*

Proof. Take $T \in \mathcal{N}_L(r, r, d_1, d_2; \alpha^1, \alpha^2)$, where $\sigma_L > \sigma_1$ therefore T is such that

$$0 \to E_2 \to E_1(D) \to \operatorname{coker}(\phi),$$

where $\operatorname{coker}(\phi)$ is a torsion sheaf. Take ϕ_0 a generic element of $\operatorname{ParHom}(E_{x,2}, E_{x,1})$. Its matrix has cokernel of some dimension biger than or equal to r_x. Hence, we estimate the value of the degree of $\operatorname{coker}(\phi)$ as bigger than or equal to the sum $\sum_{x \in D} r_x$. The formula follows from this. $\square$

Proposition 2.9.6. *If $d_1 - d_2$ is very large then $\mathcal{N}_L(r, r, d_1, d_2; \alpha^1, \alpha^2) \neq \emptyset$.*

Proof. Take E_1 and E_2 stable parabolic bundles. These bundles, together with any injective parabolic morphism $\phi : E_2 \to E_1(D)$ conform a σ-stable parabolic triple for σ large, since both bundles are stable and that for subtriples of different rank the arguments of Proposition 2.9.3 say that the subtriple does not destabilise T.

Now we show that there is always an injective morphism ϕ as above. If $d_1 - d_2$ is very large then $\operatorname{ParHom}(E_2, E_1(D))$ is a bundle of very large degree and $H^1(\operatorname{ParHom}(E_2, E_1(D)) = 0$. So there are many elements in $H^0(\operatorname{ParHom}(E_2, E_1(D))$ by Riemman-Roch. Moreover, for any $x \in X - D$,

$$\dim H^0(\operatorname{ParHom}(E_2, E_1(D))(-x)) = \dim H^0(\operatorname{ParHom}(E_2, E_1(D))) - r_1^2,$$

hence there are $\phi \in \operatorname{ParHom}(E_2, E_1(D))$ inducing a monomorphism at x. This ϕ is the injective parabolic morphism required above. $\square$

Chapter 3

Parabolic $\mathrm{U}(p,q)$-Higgs Bundles.

3.1 Parabolic $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(p,q)$-Higgs bundles.

Let E be a parabolic vector bundle over X and $D = x_1 + \cdots + x_s$ the effective divisor on X defined by a set $\{x_1, \ldots, x_s\}$ of different marked points of X.

Definition 3.1.1. A *parabolic Higgs bundle* or *parabolic* $\mathrm{GL}(n,\mathbb{C})$*-Higgs bundle* consists of a parabolic vector bundle E over X together with a strongly parabolic homomorphism

$$\Phi : E \to E \otimes K(D),$$

where $K(D)$ is the canonical bundle tensored with the line bundle defined by the divisor D.

Definition 3.1.2. A parabolic Higgs bundle (E, Φ) is said to be *stable* if $\mathrm{par}\mu(E') < \mathrm{par}\mu(E)$ for all proper parabolic subbundles which are preserved by Φ, i.e. $\Phi(E') \subset E' \otimes K(D)$. Semistability is defined by replacing the strict inequality by the weak inequality.

A parabolic Higgs bundle (E, Φ) is said to be polystable if it is a direct sum of stable parabolic Higgs bundles with the same parabolic slope.

The standard properties of stable bundles also apply to parabolic Higgs bundles, for example, if (E, Φ) and (F, Ψ) are stable parabolic Higgs bundles with the same parabolic slope, then there are no parabolic maps between them unless they are isomorphic, and the only parabolic endomorphisms of a stable parabolic Higgs bundle are the scalar multiples of the identity.

Definition 3.1.3. Let α be the system of weights of a parabolic Higgs bundle (E, Φ) we say that the weights are *generic* when every semistable parabolic Higgs bundle is automatically stable, i.e. when there are no strictly semistable parabolic Higgs bundles.

We denote $\mathcal{M}(r,d;\alpha)$ to the moduli space of stable parabolic Higgs bundles of rank r, degree a and system of weights α.

Proposition 3.1.4 ([Y1]). *The moduli space $\mathcal{M}(r,d;\alpha)$ of parabolic Higgs bundles of rank r degree d and system of weights α, is a quasiprojective complex algebraic variety of dimension*

$$r^2(2g-2)+2+\sum_{x\in D}(r^2-\sum_i m_i(x)^2).$$

Let ε be an hermitian symmetric bilinear form on $\mathbb{C}^n$ with signature (p,q) $(p+q=n)$, i.e.

$$\varepsilon(z,w)=z_1\bar{w}_1+\cdots+z_p\bar{w}_p-z_{p+1}\bar{w}_{p+1}-\cdots-z_n\bar{w}_n$$

The unitary group with signature (p,q) is defined as

$$\mathrm{U}(p,q)=\{A\in \mathrm{GL}(n,\mathbb{C}) \text{ such that } \varepsilon(Az,Aw)=\varepsilon(z,w)\}.$$

Definition 3.1.5. A parabolic $\mathrm{U}(p,q)$-*Higgs bundle* is a parabolic bundle E such that $E=V\oplus W$, where V and W are parabolic vector bundles of rank p and q respectively, and a strongly parabolic homomorphism

$$\Phi=\begin{pmatrix}0 & \beta\\ \gamma & 0\end{pmatrix}:E\to E\otimes K(D). \tag{3.1}$$

where $\beta:W\to V\otimes K(D)$ and $\gamma:V\to W\otimes K(D)$ are strongly parabolic homomorphisms.

Definition 3.1.6. We say that (E,Φ) is a stable parabolic $\mathrm{U}(p,q)$-Higgs bundle if the slope stability condition $\mathrm{par}\mu(E')<\mathrm{par}\mu(E)$, is satisfied for all Φ-invariant parabolic subbundles of the form $E'=V'\oplus W'$ i.e. for all parabolic subbundles $V'\subset V$, $W'\subset W$ such that

$$\beta:W'\to V'\otimes K(D) \tag{3.2}$$
$$\gamma:V'\to W'\otimes K(D). \tag{3.3}$$

Semistability and polystability are defined analogously to the way they are defined for parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles.

We define $\mathcal{U}(p,q,a,b,\alpha,\eta)$ to be the moduli space of stable parabolic $\mathrm{U}(p,q)$-Higgs bundles with the following topological type: ranks $\mathrm{rk}\,(V)=p$, $\mathrm{rk}\,(W)=q$, degrees $\deg(V)=a$, $\deg(W)=b$, system of weights of V denoted by α and system of weights for W denoted by η. When the topological type is fixed we will not write it.

Assumption 1. In the following we assume that the parabolic $\mathrm{U}(p,q)$-Higgs bundles E have full flags, this implies that V and W have different weights on each $x \in D$. Note that this assumption implies that the weights are generic.

Proposition 3.1.7. *Let $\mathcal{M}(p+q, a+b; \alpha \cup \eta)$ be the moduli of parabolic Higgs bundles of degree $d = a+b$, rank $r = p+q$ and system of weights $\alpha\cup\eta$ (see 1.1.20) then $\mathcal{U}(p,q,a,b;\alpha,\eta)$ embeds as a closed subvariety in $\mathcal{M}$.*

Proof. The proof is similar to that in the non parabolic case (see Proposition 3.11 in [BGG]) we just need to observe that in the case $p = q$, the parabolic bundles V and W can not be parabolically isomorphic since they have different weights. $\square$

3.2 Deformations of parabolic $\mathrm{U}(p,q)$-Higgs bundles

We adapt the results for parabolic Higgs bundles in [GGM] and ordinary $U(p,q)$-Higgs bundles in [BGG] to describe the deformation theory of $\mathrm{U}(p,q)$-parabolic Higgs bundles.

Let $\mathbf{E} = (E, \Phi)$ be a $\mathrm{U}(p,q)$-parabolic Higgs bundle. Consider the following complex of sheaves:

$$C^\bullet(\mathbf{E}) : \mathrm{ParEnd}\,(V)\oplus\mathrm{ParEnd}\,(W) \xrightarrow{\mathrm{ad}\,(\Phi)} (\mathrm{SParHom}\,(V,W)\oplus\mathrm{SParHom}\,(W,V))\otimes K(D) \quad (3.4)$$

where $\mathrm{ad}\,(\Phi)(f,g) = [\Phi,(f,g)] = ((g\otimes 1_{K(D)})\gamma - \gamma f, (f\otimes 1_{K(D)})\beta - \beta g)$.

Proposition 3.2.1. *(i) The space of infinitesimal deformations of (E,Φ) is isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet(\mathbf{E}))$.*

(ii) There is a long exact sequence:

$$\begin{aligned} 0 &\to \mathbb{H}^0 \to \mathrm{H}^0(\mathrm{ParEnd}\,(V)\oplus\mathrm{ParEnd}\,(W)) \\ &\to \mathrm{H}^0((\mathrm{SParHom}\,(V,W)\oplus\mathrm{SParHom}\,(W,V))\otimes K(D)) \\ &\to \mathbb{H}^1 \to \mathrm{H}^1(\mathrm{ParEnd}\,(V)\oplus\mathrm{ParEnd}\,(W)) \\ &\to \mathrm{H}^1((\mathrm{SParHom}\,(V,W)\oplus\mathrm{SParHom}\,(W,V))\otimes K(D)) \\ &\to \mathbb{H}^2 \to 0, \end{aligned} \quad (3.5)$$

where $\mathbb{H}^i = \mathbb{H}^i(C^\bullet(\mathbf{E}))$.

(iii) If $\mathbb{H}^2(C^\bullet(\mathbf{E})) = 0$ then the moduli space is smooth in a neighbourhood of $\mathbf{E}$.

$\square$

Proposition 3.2.2. *Let* $\mathbf{E}$ *be a stable* $\mathrm{U}(p,q)$*-parabolic Higgs bundle. Then* $\mathbb{H}^0(C^\bullet(\mathbf{E})) = \mathbb{C}$ *and* $\mathbb{H}^2(C^\bullet(\mathbf{E})) = 0$.

Proof. We prove first that $\mathbb{H}^2(C^\bullet(\mathbf{E})) = 0$. By the long exact sequence in (3.5), $\mathbb{H}^2(C^\bullet(\mathbf{E})) = 0$ is equivalent to the surjectivity of $\mathrm{H}^1(\mathrm{ParEnd}\,(V)\oplus\mathrm{ParEnd}\,(W)) \to \mathrm{H}^1((\mathrm{SParHom}\,(V,W)\oplus \mathrm{SParHom}\,(W,V)) \otimes K(D))$. Using Serre Theorem for parabolic bundles (1.1.17), this is equivalent to the injectivity of

$$\begin{array}{rcl} \mathrm{H}^0(\mathrm{ParHom}\,(V,W) \oplus \mathrm{ParHom}\,(W,V)) & \to & \mathrm{H}^0((\mathrm{SParEnd}\,(V) \oplus \mathrm{SParEnd}\,(W)) \otimes K(D)) \\ (\varphi,\psi) & \mapsto & ((\psi\otimes 1_{K(D)})\gamma - \beta\varphi, (\varphi\otimes 1_{K(D)})\beta - \gamma\psi). \end{array}$$

Let (φ,ψ) such that

$$(\psi\otimes 1_{K(D)})\gamma - \beta\varphi = 0, \qquad (\varphi\otimes 1_{K(D)})\beta - \gamma\psi = 0. \tag{3.6}$$

Consider $N = \ker(\varphi)\oplus\ker(\psi) \subset V\oplus W$ and let $I = \mathrm{Im}\,(\psi)\oplus\mathrm{Im}\,(\varphi)$ be the parabolic image sheaf. Denote by $\tilde{I} \subset W \oplus V$ its parabolic saturation (i.e. take the saturation of I as a subbundle, and endow it with the induced parabolic structure). It follows from (3.6) that N and $\tilde{I}$ are parabolic Higgs subbundles of E. But since

$$0 \to N \to E \to I \to 0$$

we have that

$$\mathrm{par}\mu\,(E) = \mathrm{par}\mu\,(I) + \mathrm{par}\mu\,(N).$$

This contradicts the stability of (E,Φ), which implies that

$$\mathrm{par}\mu\,(N) < \mathrm{par}\mu\,(E)$$

and

$$\mathrm{par}\mu\,(I) \leq \mathrm{par}\mu\,(\tilde{I}) < \mathrm{par}\mu\,(E),$$

unless $N = 0$ or $I = 0$. If $N = 0$ then $\varphi : V \to W$ and $\psi : W \to V$ are both injective. Therefore V and W are isomorphic as parabolic bundles, which is impossible since they have different weights. Hence $I = 0$ and then $(\varphi,\psi) = (0,0)$.

To prove that $\mathbb{H}^0(C^\bullet(\mathbf{E})) = \mathbb{C}$ we consider again the long exact sequence in Proposition 3.2.1. So we only have to compute the kernel of

$$\mathrm{H}^0(\mathrm{ParEnd}\,(V) \oplus \mathrm{ParEnd}\,(W)) \xrightarrow{\sigma} \mathrm{H}^0((\mathrm{SParHom}\,(V,W) \oplus \mathrm{SParHom}\,(W,V)) \otimes K(D)),$$

where $\sigma(\varphi,\psi)=(\gamma\varphi-(\psi\otimes 1_{K(D)})\gamma, \beta\psi-(\varphi\otimes 1_{K(D)})\beta)$. Clearly the scalar multiples of the identity $(\lambda\cdot 1_V, \lambda\cdot 1_W)$ are in $\ker(\sigma)$. Take now $(\varphi,\psi)\in\ker(\sigma)$, i.e.

$$\gamma\varphi-(\psi\otimes 1_{K(D)})\gamma=0, \qquad \beta\psi-(\varphi\otimes 1_{K(D)})\beta=0. \tag{3.7}$$

Let $N=\ker(\varphi)\oplus\ker(\psi)\subset V\oplus W$ the direct sum of the kernels of φ and ψ, and let $I=\mathrm{Im}\,(\psi)\oplus\mathrm{Im}\,(\varphi)$ be the parabolic image sheaf. Denote by $\tilde{I}\subset W\oplus V$ its parabolic saturation. It follows from (3.7) that N and $\tilde{I}$ are parabolic Higgs subbundles of E. The fact that $\mathrm{par}\mu\,(E)=\mathrm{par}\mu\,(N)+\mathrm{par}\mu\,(I)$ and the stability of (E,Φ) imply that either $N=0$ or $I=0$. Suppose that $I\neq 0$, then take $\lambda\in\mathbb{C}$ an eigenvalue of φ. So $(\varphi-\lambda\cdot 1_V, \psi-\lambda\cdot 1_W)$ is in $\ker(\sigma)$ that is, it must be zero and hence $(\varphi,\psi)=(\lambda\cdot 1_V,\lambda\cdot 1_W)$. □

3.3 Dimension of the moduli space.

Proposition 3.3.1. *For full flags, the moduli space $\mathcal{U}$ of stable $U(p,q)$-parabolic Higgs bundles is a smooth complex variety of complex dimension*

$$1+(g-1)(p+q)^2+s\frac{(p+q)^2-(p+q)}{2},$$

where g is the genus of X, and s is the number of marked points.

Proof. Smoothness follows from Proposition 3.2.2.

$$\begin{aligned}
\dim_{\mathbb{C}}\mathcal{U} &= \dim\mathbb{H}^1(C^\bullet(\mathbf{E}))=1-\chi(\mathrm{ParEnd}\,(V)\oplus\mathrm{ParEnd}\,(W))\\
&+ \chi((\mathrm{SParHom}\,(V,W)\oplus\mathrm{SParHom}\,(W,V))\otimes K(D))\\
&= 1-(p^2+q^2)(1-g)-\deg(\mathrm{ParEnd}\,(V))-\deg(\mathrm{ParEnd}\,(W))+2pq((1-g)\\
&\quad +\deg(\mathrm{SParHom}\,(V,W)+\deg(\mathrm{SParHom}\,(W,V))+2pq(2g-2)+2pqs\\
&= 1+(g-1)(p+q)^2+2pqs+(p^2+q^2-2pq)s\\
&\quad +\sum_{x\in D}(\dim\mathrm{SParHom}\,(V,W)_x+\dim\mathrm{SParHom}\,(W,V)_x-\dim\mathrm{ParEnd}\,(V)_x\\
&- \dim\mathrm{ParEnd}\,(W)_x\\
&= 1+(g-1)(p+q)^2+\frac{s}{2}((p+q)^2-(p+q)),
\end{aligned}$$

where we have used that

$$\begin{aligned}
\deg(\mathrm{ParHom}\,(V,W)) &= p\deg(W)-q\deg(V)+\sum_{p\in D}(\dim\mathrm{ParHom}\,(V_x,W_x)-pq)\\
\deg(\mathrm{SParHom}\,(V,W)) &= p\deg(W)-q\deg(V)+\sum_{p\in D}(\dim\mathrm{SParHom}\,(V_x,W_x)-pq),
\end{aligned}$$

from the following short exact sequence,

$$0 \to \mathrm{ParHom}\,(V,W) \to \mathrm{Hom}\,(V,W) \to \bigoplus_{x\in D} \frac{\mathrm{Hom}\,(V_x,W_x)}{\mathrm{ParHom}\,(V_x,W_x)} \to 0$$

and,

$$\begin{aligned}\dim \mathrm{SParHom}\,(V,W)_x + \dim \mathrm{SParHom}\,(W,V)_x &= \\ = \dim \mathrm{ParEnd}\,(V\oplus W)_x - \dim \mathrm{ParEnd}\,(V)_x &- \dim \mathrm{ParEnd}\,(W)_x = pq.\end{aligned}$$

We are using full flags and different weights, this allow us to compute the dimensions of these vector spaces of parabolic homomorphisms. □

Remark 3.3.2. For $s = 0$ we recover the formula for the non parabolic case given in [BGG]. As expected, this dimension is half the dimension of the moduli of parabolic Higgs bundles of rank $p+q$.

Remark 3.3.3. When $g = 0$ we need $s \geq 3$ in order to have a non empty moduli space.

Remark 3.3.4. Note that $\dim_{\mathbb{C}} \mathcal{U}$ is half the dimension of $\mathcal{M}$, and note also that the dimension of $\mathcal{U}$ depends on the sum $p+q$.

3.4 Parabolic Toledo invariant

In analogy with the non-parabolic case [BGG], one can associate a Toledo invariant to a parabolic $\mathrm{U}(p,q)$-Higgs bundle.

Definition 3.4.1. The *parabolic Toledo invariant* corresponding to the parabolic Higgs bundle $(E = V \oplus W, \Phi)$ is defined as

$$\tau = 2\,\frac{q\,\mathrm{pardeg}\,(V) - p\,\mathrm{pardeg}\,(W)}{p+q}. \tag{3.8}$$

The Toledo invariant will give us a way of classifying components of the moduli space of $U(p,q)$-parabolic Higgs bundles. So we first determine the possible values that it can take.

Proposition 3.4.2. *Let $(E = V\oplus W, \Phi = (\beta,\gamma))$ be a stable parabolic $\mathrm{U}(p,q)$-Higgs bundle. Then*

$$p(\mathrm{par}\mu\,(V) - \mathrm{par}\mu\,(E)) < \mathrm{rk}\,(\gamma)\big(g-1+\frac{s}{2}\big)$$

$$q(\mathrm{par}\mu\,(W) - \mathrm{par}\mu\,(E)) < \mathrm{rk}\,(\beta)\big(g-1+\frac{s}{2}\big).$$

Proof. Consider the parabolic bundles $N = \ker(\gamma)$ and $I = \mathrm{Im}\,(\gamma) \otimes K(D)^{-1}$. We have an exact sequence of parabolic bundles

$$0 \to N \to V \to I \otimes K(D) \to 0$$

and

$$\begin{aligned} \operatorname{pardeg}(V) &= \operatorname{pardeg}(N) + \operatorname{pardeg}(I \otimes K(D)) \\ &= \operatorname{pardeg}(N) + \operatorname{pardeg}(I) + \operatorname{rk}(I)(2g-2+s). \end{aligned} \tag{3.9}$$

Note that I is a subsheaf of W and the map $I \hookrightarrow W$ is a parabolic map. Let $\tilde{I} \subset W$ be its saturation, which is a subbundle of W, and endow it with the induced parabolic structure. So N and $V \oplus \tilde{I} \subset E$ are Φ-invariant parabolic subbundles of E. The stability of (E, Φ) implies that

$$\begin{aligned} \operatorname{par}\mu\,(N) < &\ \operatorname{par}\mu\,(E) \\ \operatorname{par}\mu\,(V \oplus I) \leq \operatorname{par}\mu\,(V \oplus \tilde{I}) < &\ \operatorname{par}\mu\,(E). \end{aligned} \tag{3.10}$$

This yields

$$\begin{aligned} \operatorname{pardeg}(N) &< \operatorname{rk}(N)\operatorname{par}\mu\,(E) \\ \operatorname{pardeg}(V) + \operatorname{pardeg}(I) &< (p + \operatorname{rk}(I))\operatorname{par}\mu\,(E) \end{aligned}$$

Adding both inequalities and using (3.9) we obtain

$$2\operatorname{pardeg}(V) < 2p\operatorname{par}\mu\,(E) + \operatorname{rk}(I)(2g-2+s),$$

and hence,

$$p(\operatorname{par}\mu\,(V) - \operatorname{par}\mu\,(E)) < \operatorname{rk}(\gamma)\left(g - 1 + \frac{s}{2}\right).$$

The proof of the other inequality in Proposition is analogous. $\square$

One has the following bound for the Toledo invariant.

Proposition 3.4.3. *Let (E, Φ) be a semistable $U(p,q)$-parabolic Higgs subbundle. Then,*

$$|\tau| < \min\{p, q\}(2g-2+s),$$

Proof. Using that

$$\operatorname{par}\mu\,(E) = \frac{p}{p+q}\operatorname{par}\mu\,(V) + \frac{q}{p+q}\operatorname{par}\mu\,(W), \tag{3.11}$$

Proposition 3.4.2 may be rewritten as

$$\begin{aligned} q(\operatorname{par}\mu\,(E) - \operatorname{par}\mu\,(W)) &< \operatorname{rk}(\gamma)(g - 1 + \frac{s}{2}) \\ p(\operatorname{par}\mu\,(E) - \operatorname{par}\mu\,(V)) &< \operatorname{rk}(\beta)(g - 1 + \frac{s}{2}). \end{aligned}$$

By (3.11) we also have $\tau = 2p(\operatorname{par}\mu\,(V) - \operatorname{par}\mu\,(E)) = 2q(\operatorname{par}\mu\,(E) - \operatorname{par}\mu\,(W))$. The result follows. $\square$

3.5 Moduli space and Gauge theory.

In order to study the topology of $\mathcal{U}$ we need a gauge-theoretic interpretation of this moduli space in terms of solutions of the Hitchin equations. One can adapt the arguments given by Simpson [S2] for the case of parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles to the $\mathrm{U}(p,q)$ situation, along the lines of what is done in [BGG] in the non-parabolic case. Similarly, to construct the moduli space from this point of view, one can adapt the construction given by Konno [K] (see also [NaSt]) in the parabolic $\mathrm{GL}(n,\mathbb{C})$ case. We thus give here only a scketch of this.

A parabolic structure on a smooth vector bundle is defined in a similar way to what is done in the holomorphic category. Let E be a smooth parabolic vector bundle of rank n and fix a hermitian metric h in E which is smooth in $X \setminus D$ and whose (degenerate) behaviour around the marked points is given as follows. We say that a local frame $\{e_1, \dots, e_n\}$ for E around x *respects the flag at* x if $E_{x,i}$ is spanned by the vectors $\{e_{M_i+1}(x), \dots, e_n(x)\}$, where $M_i = \sum_{j\le i} m_j$. Let z be a local coordinate around x such that $z(x) = 0$. We require that h be of the form

$$h = \begin{pmatrix} |z|^{2\tilde{\alpha}_1} & & 0 \\ & \ddots & \\ 0 & & |z|^{2\tilde{\alpha}_n} \end{pmatrix}$$

with respect to some local frame around x which respects the flag at x.

A unitary connection d_A associated to a smooth $\bar{\partial}$ operator $\bar{\partial}_E$ on E via the hermitian metric h is singular at the marked points: if we write $z = \rho \exp(\sqrt{-1}\theta)$ and $\{e_i\}$ is the local frame used in the definition of h, then with respect to the local frame $\{\epsilon_i = e_i/|z|^{\tilde{\alpha}_i}\}$, the connection is of the form

$$d_A = d + \sqrt{-1}\begin{pmatrix} \tilde{\alpha}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{\alpha}_r \end{pmatrix} d\theta + A',$$

where A' is regular. We denote the space of smooth $\bar{\partial}$-operators on E by $\mathscr{C}_E$, the space of associated h-unitary connections by $\mathscr{A}_E$, the group of complex parabolic gauge transformations by $\mathscr{G}_E^{\mathbb{C}}$ and the subgroup of h-unitary parabolic gauge transformations by $\mathscr{G}_E$.

Let V and W be smooth parabolic vector bundles equipped with hermitian metrics h_V and h_W adapted to the parabolic structures in the sense explained above. We denote $\mathscr{C} := \mathscr{C}_V \times \mathscr{C}_W$, $\mathscr{G}^{\mathbb{C}} := \mathscr{G}_V^{\mathbb{C}} \times \mathscr{G}_W^{\mathbb{C}}$, $\mathscr{G} := \mathscr{G}_V \times \mathscr{G}_W$. The space of Higgs fields is $\mathbf{\Omega} = \mathbf{\Omega}^+ \oplus \mathbf{\Omega}^-$, where $\mathbf{\Omega}^+ = \Omega^{1,0}(\mathrm{SParHom}\,(W,V) \otimes \mathcal{O}(D))$, $\mathbf{\Omega}^- = \Omega^{1,0}(\mathrm{SParHom}\,(V,W) \otimes \mathcal{O}(D))$. Here we regard $\mathrm{SParHom}\,(W,V)$ and $\mathrm{SParHom}\,(V,W)$ as smooth vector bundles defined as in the holomorphic category.

Following Biquard [B] and Konno [K], we introduce certain weighted Sobolev norms and denote the corresponding Sobolev completions of the spaces defined above by $\mathscr{C}_1^p$, $\mathbf{\Omega}_1^p$, $(\mathscr{G}^{\mathbb{C}})_2^p$ and $\mathscr{G}_2^p$. Let

$$\mathscr{H} = \{(\bar\partial_E, \Phi) \in \mathscr{C} \times \mathbf{\Omega} \mid \bar\partial_E \Phi = 0\}$$

and let $\mathscr{H}_1^p$ be the corresponding subspace of $\mathscr{C}_1^p \times \mathbf{\Omega}_1^p$.

Let $\bar\partial_E = (\bar\partial_V, \bar\partial_W)$ with $\bar\partial_V \in \mathscr{C}_V$ and $\bar\partial_W \in \mathscr{C}_W$, and $\Phi = (\beta, \gamma)$ with $\beta \in \mathbf{\Omega}^+$ and $\gamma \in \mathbf{\Omega}^-$. Let $F(A_V)$ and $F(A_W)$ be the curvatures of the h_V and h_W unitary connections corresponding to $\bar\partial_V$ and $\bar\partial_W$, respectively. Let β^* and γ^* be the adjoints with respect to h_V and h_W. Fix a Kähler form on X with volume of X normalized to 2π. We consider the moduli space $\mathcal{S}$ defined by the subspace of elements in $\mathscr{H}_1^p$ satisfying *Hitchin's equations*

$$F(A_V) + \beta\beta^* + \gamma^*\gamma = -\sqrt{-1}\mu \operatorname{Id}_V \omega\ , \tag{3.12}$$

$$F(A_W) + \gamma\gamma^* + \beta^*\beta = -\sqrt{-1}\mu \operatorname{Id}_W \omega\ , \tag{3.13}$$

modulo gauge transformation in $\mathscr{G}_2^p$, where the equation is only defined on $X \setminus D$. Taking the trace of the equations, adding them, integrating over $X \setminus D$, and using the Chern-Weil formula for parabolic bundles, we find that $\mu = \operatorname{par}\mu\,(V \oplus W)$.

The subspace of smooth points in $\mathscr{H}_1^p$ carries a Kähler metric induced by the complex structure of X and the hermitian metrics h_V and h_W. The Hitchin equations are moment map equations for the action of $\mathscr{G}_2^p$ on this subspace. In particular, the smooth part of $\mathcal{S}$, which corresponds to irreducible solutions, is obtained as a Kähler quotient. Under the genericity assumptions on the parabolic weights, all the solutions are irreducible and the moduli space $\mathcal{S}$ is a smooth Kähler manifold.

Fix the topological invariants $p = \operatorname{rk} V$, $q = \operatorname{rk} W$, $a = \deg V$, $b = \deg W$ and the weight types α and η of V and W, respectively. Then

$$\mathcal{U}(p, q, a, b; \alpha, \eta) \cong (\mathscr{H}^{stable})_1^p / (\mathscr{G}^{\mathbb{C}})_2^p.$$

Moreover, if $\mathcal{S}(p, q, a, b; \alpha, \eta)$ is the moduli space of solutions for these fixed invariants, we have the following.

Theorem 3.5.1. *There is a homeomorphism*

$$\mathcal{U}(p, q, a, b; \alpha, \eta) = \mathcal{S}(p, q, a, b; \alpha, \eta).$$

3.6 Parabolic $\mathrm{U}(p, q)$-Higgs bundles and representations.

Let X be a compact Riemann surface of genus $g \geq 0$ and let $S = \{x_1, \ldots, x_s\}$ be a set of distinct points of X. Let $\Gamma = \pi_1(X \setminus S)$ be the fundamental group of $X \setminus S$. The group Γ is generated by the usual generators a_i, b_i, $1 \leq i \leq g$ of $\pi_1(X)$, together with additional generators $\gamma_1, \ldots \gamma_s$ corresponding to loops enclosing each x_i simply, not enclosing any x_j, $j \neq i$, and which are homotopic to zero relatively to the base point on X. There is also the relation $[a_1, b_1] \ldots [a_g, b_g]\gamma_1 \ldots \gamma_s = 1$, where $[a_i, b_i]$ is the commutator of a_i and b_i.

Parabolic Higgs bundles are related to representations of Γ. To be precise, let us fix integers $n = \operatorname{rk} E$, $d = \deg E$ and the weight type $\alpha = \{\alpha(x)\}_{x \in S}$, where $\alpha(x) = (\alpha_1(x), \ldots, \alpha_{r(x)}(x))$ are weights with multiplicities $k_i(x)$ for every $x \in S$. It is convenient to repeat each weight according to its multiplicity, by setting $\tilde{\alpha}_1(x) = \ldots = \tilde{\alpha}_{k_1(x)}(x) = \alpha_1(x)$, etc., thus having weights $0 \leq \tilde{\alpha}_1(x) < \ldots < \tilde{\alpha}_n(x) < 1$.

For every $x \in S$ there is a matrix $C_i \in \mathrm{U}(n)$ defined by

$$C_i = \begin{pmatrix} \exp(2\pi\sqrt{-1}\tilde{\alpha}_1(x)) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi\sqrt{-1}\tilde{\alpha}_n(x)) \end{pmatrix}. \tag{3.14}$$

Consider the set of representations $\mathrm{Hom}_{\alpha}(\Gamma, \mathrm{GL}(n, \mathbb{C}))$ defined by semisimple homomorphisms $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ such that $\rho(\gamma_i)$ is conjugated to C_i by an element in $\mathrm{GL}(n, \mathbb{C})$ for $1 \leq i \leq s$. Here by semisimple we mean that ρ is a direct sum of irreducible representations. The moduli space of representations of Γ in $\mathrm{GL}(n, \mathbb{C})$ with fixed holonomy $[C_i]$ is defined by the quotient

$$\mathcal{R}(n; \alpha) := \frac{\mathrm{Hom}_{\alpha}(\Gamma, \mathrm{GL}(n, \mathbb{C}))}{\mathrm{GL}(n, \mathbb{C})},$$

where $\mathrm{GL}(n, \mathbb{C})$ acts by conjugation. The set $\mathcal{R}(n, \alpha)$ has a natural structure of a complex algebraic variety. The following is proved by Simpson in [S2].

Theorem 3.6.1. *Let (n, d, α) be such that*

$$d + \sum_{x \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_n(x)) = 0,$$

i.e., the parabolic degree vanishes. Then there is a homeomorphism

$$\mathcal{R}(n; \alpha) \cong \mathcal{M}(n, d; \alpha).$$

This generalizes the Theorem of Metha–Seshadri [MS] which identifies the moduli space of parabolic bundles of type (n, d, α) with vanishing parabolic degree with the moduli space of representations of Γ in $\mathrm{U}(n)$ with fixed holonomy conjugated to C_i around the marked points.

There is a similar correspondence between representations of Γ in $\mathrm{U}(p,q)$ and parabolic $\mathrm{U}(p,q)$-Higgs bundles. To explain this, let us come back to the notation in Section 3.1 and fix the types of the parabolic bundles V and W to be (p, a, α) and (q, b, η), respectively. For every $x_i \in S$ there are matrices $C_i \in \mathrm{U}(p)$ and $C_i' \in \mathrm{U}(q)$ defined as in (3.14) by the weight systems α and η, respectively.

Consider now the set of representations $\mathrm{Hom}_{\alpha,\eta}(\Gamma, \mathrm{U}(p,q))$ defined by semisimple homomorphisms $\rho : \Gamma \to \mathrm{U}(p,q)$ such that $\rho(\gamma_i)$ is conjugated to $C_i \times C_i' \in \mathrm{U}(p) \times \mathrm{U}(q)$ — maximal compact subgroup of $\mathrm{U}(p,q)$ — by an element in $\mathrm{U}(p,q)$ for $1 \leq i \leq s$. Define the moduli space of representations of Γ in $\mathrm{U}(p,q)$ with fixed holonomy $\mathrm{U}(p,q)$-conjugated to $C_i \times C_i'$ by the quotient

$$\mathcal{R}(p,q;\alpha,\eta) := \frac{\mathrm{Hom}_{\alpha,\eta}(\Gamma, \mathrm{U}(p,q))}{\mathrm{U}(p,q)}.$$

We can adapt the arguments of Simpson [S2] to prove the following.

Theorem 3.6.2. *Let (p, a, α) and (q, b, η) be such that* $\mathrm{pardeg}\,(V) + \mathrm{pardeg}\,(W) = 0$. *Then there is a homeomorphism*

$$\mathcal{R}(p,q;\alpha,\eta) \cong \bigsqcup_{a,b} \mathcal{U}(p,q,a,b;\alpha,\eta).$$

Note that $(p, q, a, b, \alpha, \eta)$ must also satisfy the Milnor–Wood inequality, which in this cases reduces to

$$\mathrm{pardeg}\,(V) \leq \min\{p,q\}(g - 1 + s/2).$$

Theorem 3.6.3. *Let X be a compact Riemann surface and $S = \{x_i\}_{i=1}^s$ a finite set of marked points on X. Let $\mathcal{R}(p,q;\alpha,\eta)$ be the moduli space of representations of $\pi_1(X - S)$ in $\mathrm{U}(p,q)$ with fixed holonomy classes in $\mathrm{U}(p) \times \mathrm{U}(q)$. For α and η generic weights, the number of non-empty connected components of $\mathcal{R}(n,1;\alpha,\eta)$ equals the number of integers a such that*

$$a + \sum_{x \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_p(x)) \leq \tau_L/2,$$

where τ_L is given by (3.24).

Like in the proof of Theorem 3.6.1 ([S2]), the main ingredients in the proof of Theorem 3.6.2 are, on the one hand, the correspondence given by Theorem 3.5.1 between polystable parabolic $\mathrm{U}(p,q)$-Higgs bundles and solutions to Hitchin equations, and, on the other, the existence of a harmonic adapted metric on a $\mathrm{U}(p,q)$-bundle with a semisimple meromorphic flat connection with simple poles. To explain this, let us come back to the framework of Section 3.5, and consider smooth parabolic vector bundles V and W of type (p,a,α) and (q,b,η), respectively. On the bundle $V \oplus W$ we consider flat $\mathrm{U}(p,q)$-connections D on $X \setminus S$, meromorphic at $x_i \in S$ and whose residue at x_i is conjugated to $C_i \times C_i'$. These connections are in correspondence with elements in $\mathrm{Hom}_{\alpha,\eta}(\Gamma, \mathrm{U}(p,q))$. We say that D is semisimple if the corresponding representation is semisimple.

Let $h = (h_V, h_W)$, where h_V and h_W are adapted hermitian metrics on V and W, respectively. We decompose D as $D = d_A + \Psi$, where d_A is a $\mathrm{U}(p) \times \mathrm{U}(q)$ connection and Ψ takes values in $\mathfrak{m}$, where $\mathfrak{u}(p,q) = \mathfrak{u}(p) \oplus \mathfrak{u}(q) + \mathfrak{m}$ is the Cartan decomposition of the Lie algebra of $\mathrm{U}(p,q)$. We say that h is harmonic if $d_A^* \Psi = 0$. Then we have the following ([C],[S2])

Theorem 3.6.4. *A connection D as above is semisimple if and only if there exists a harmonic hermitian metric $h = (h_v, h_W)$.*

The relation with parabolic $\mathrm{U}(p,q)$-Higgs bundle is given as follows. If D is semisimple flat connection as above and h is a harmonic solution, then the pair (d_A, Φ), where Φ is determined by the equation $\Psi = \Phi + \Phi^*$, solves the $\mathrm{U}(p,q)$-Hitchin equations and hence, by Theorem 3.5.1, corresponds to a polystable parabolic $\mathrm{U}(p,q)$-Higgs bundle. Conversely if we have a polystable parabolic $\mathrm{U}(p,q)$-Higgs bundle we can find a solution (d_A, Φ) to the Hitchin equations and then out of it a solution to the harmonic equation on the flat connection $D = d_A + \Phi + \Phi^*$, which is then semisimple by Theorem 3.6.4.

Definition 3.6.5. We call the α and η *generic* weights for the representation if they are generic weights for the bundle associated to it.

3.7 Topology of the moduli space.

To study the topology of the moduli space $\mathcal{U}$, we follow the work done for the non-parabolic case in [BGG] and we use Hitchin's Morse-theoretic approach to the problem.

Bott-Morse theory has been used already in the context of parabolic Higgs bundles in [GGM, BY, NS]. By theorem 3.5.1 we have an action of S^1 on $\mathcal{U}$ given by

$$\begin{array}{rcl} \psi: \quad S^1 \times \mathcal{U} & \to & \mathcal{U} \\ (\lambda, (E, \Phi)) & \mapsto & (E, \lambda\Phi). \end{array}$$

The identification $\mathcal{U}(p,q,a,b;\alpha,\eta) = \mathcal{S}(p,q,a,b;\alpha,\eta)$ respects the circle action so, with respect to the complex structure on $\mathcal{U}(p,q,a,b;\alpha,\eta)$, this is a Hamiltonian action of the circle on $\mathcal{S}(p,q,a,b;\alpha,\eta)$, with associated moment map

$$[(E,\Phi)] \mapsto -\frac{1}{2}\|\Phi\|^2 = -\sqrt{-1}\int_X \operatorname{Tr}(\Phi\Phi^*).$$

In our situation, we shall use as Bott-Morse function the positive function, $f : \mathcal{U} \to \mathbb{R}$ defined as

$$f([E,\Phi]) = \|\Phi\|^2. \tag{3.15}$$

Clearly f is bounded below since it is non-negative and is proper. Since $\mathcal{U} \subset \mathcal{M}$ is a closed subset, this follows from the properness of the moment map associated to the circle action on $\mathcal{M}$ [Bi] (see also [GGM]).

To prove the connectedness of $\mathcal{U}$, we use the following standard result from general topology.

Proposition 3.7.1. *If Z is a Hausdorff space and $f : Z \to \mathbb{R}$ is proper and bounded below then f attains a minimum on each connected component of Z. Therefore, if the subspace of local minima of f is connected then so is Z.*

Corollary 3.7.2. *The function $f : \mathcal{U} \to \mathbb{R}$ defined in (3.15) has a minimum on each connected component of $\mathcal{U}$. Moreover, if the subspace of local minima of f is connected then so is $\mathcal{U}$.*

Now we will describe the minima of f. For this we introduce the subset of $\mathcal{U}$ defined by

$$\mathcal{N} = \mathcal{N}(p,q,a,b;\alpha,\eta) = \{(E,\Phi) \in \mathcal{U}(p,q,a,b;\alpha,\eta) \text{ such that } \beta = 0 \text{ or } \gamma = 0\}.$$

Proposition 3.7.3. *For every $(E,\Phi) \in \mathcal{U}$*

$$f(E,\Phi) \geq \frac{|\tau|}{2}$$

with equality if and only if $(E,\Phi) \in \mathcal{N}$.

Proof. The proof goes exactly the same way as in [BGG] except that we are using adapted metrics on the bundles V and W.

Taking trace, integrating over X and using Proposition 1.4.8 on Hitchin's equations (3.12), we have

$$\operatorname{pardeg}(V) = \mu(E) - \frac{1}{\pi}\|\beta\| + \frac{1}{\pi}\|\gamma\|,$$

which is equivalent to

$$\frac{1}{\pi}\|\beta\| - \frac{1}{\pi}\|\gamma\| = p(\operatorname{par}\mu(E) - \operatorname{par}\mu(V)) = -\frac{\tau}{2}.$$

Since $f(E,\Phi) = \|\Phi\| = \frac{1}{\pi}\|\beta\| + \frac{1}{\pi}\|\gamma\|$, it follows that

$$f(E,\Phi) = \begin{cases} \frac{2}{\pi}\|\gamma\| - \frac{\tau}{2} & \text{whenever } \tau < 0 \\ \frac{2}{\pi}\|\beta\| + \frac{\tau}{2} & \text{whenever } \tau > 0, \end{cases} \tag{3.16}$$

thus giving the result. □

We will prove that $\mathcal{N}$ is the subvariety of local minima of f. To do this we have to describe the critical points of f and characterise the local minima. By a theorem of Frankel [F], the critical points of f are exactly the fixed points of the circle action.

For a fixed point (E,Φ) of the circle action, we have an isomorphism $(E,\Phi) \cong (E, e^{\sqrt{-1}\theta}\Phi)$ which yields the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E\otimes K(D) \\ {\scriptstyle \psi_\theta}\downarrow & & \downarrow{\scriptstyle \psi_\theta\otimes 1_{K(D)}} \\ E & \xrightarrow{e^{\sqrt{-1}\theta}\Phi} & E\otimes K(D). \end{array}$$

Proposition 3.7.4 (C. Simpson, [S2], Thm.8). *The equivalence class of a stable parabolic Higgs bundle (E,Φ) is fixed under the action of S^1 if and only if it is a* parabolic Hodge *bundle. This means that E has a direct sum decomposition*

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$$

as parabolic bundles, such that $\Phi_l = \Phi|_{E_l}$ belongs to $H^0(\operatorname{SParHom}(E_l, E_{l+1}) \otimes K(D))$. If $\Phi_l \neq 0$, then the weight of the isomorphism $\psi : E \longrightarrow E$ on E_{l+1} increases by one.

The decomposition of E is given by the eigenbundles corresponding to the eigenvalues of the circle action on (E,Φ).

Corollary 3.7.5. *Under the hypothesis of the above theorem. If (E, Φ) is stable, each Φ_l is non zero and the E_l are alternately contained in V and W.*

Proof. Suppose there is an l such that $\Phi_l = 0$ then E_l is a Φ-invariant subbundle of E. Thus we can split E into two Φ-invariant subbundles

$$E_0 \oplus \cdots \oplus E_l \quad \text{and} \quad E_{l+1} \oplus \cdots \oplus E_m$$

each of them can be written as subbundle or as a quotient of E, hence by stability of E we get that $\text{par}\mu\,(E_l \oplus \cdots \oplus E_m)$ is bigger than $\text{par}\mu\,(E)$. This give us a contradiction with the stability of E if we consider it as a subbundle instead of considering it as a quotient. In order to prove the second statement lets consider $E_l \cap V \neq \{0\}$ and $E_l \cap W \neq \{0\}$.

Hence we can decompose E_l into $E_l \cap V \oplus E_l \cap W$. In order to simplify notation we write

$$E_l = E_l^V \oplus E_l^W. \tag{3.17}$$

Recall that

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where $\gamma : V \to W \otimes K(D)$ and $\beta : W \to V \otimes K(D)$. Therefore the decomposition of E_l described in (3.17) gives us

$$\gamma(E_l^V) \subset E_{l+1}^W \otimes K(D)$$
$$\beta(E_l^W) \subset E_{l+1}^V \otimes K(D).$$

Hence we get the following Φ-invariant decomposition of E

$$E = (E_0^V \oplus E_1^W \oplus \cdots \oplus E^V) \oplus (E_0^W \oplus E_1^V \oplus \cdots \oplus E_m^W).$$

That is $E_0^V \oplus E_1^W \oplus \cdots \oplus E^V$ and $E_0^W \oplus E_1^V \oplus \cdots \oplus E_m^W$ are two Φ-invariant subbundles of E contradicting the stability of (E, Φ). $\square$

Now we will compute the index of a critical point (E, Φ). To do this we need to write the complex in (3.4) in terms of the eigenbundle decomposition provided by Proposition 3.7.4. We consider without loss of generality $E_0, \ldots, E_m \subset V$ and $E_1, \ldots, E_{m-1} \subset W$. Hence

$$\text{ParEnd}\,(V) \oplus \text{ParEnd}\,(W) = \bigoplus_{-m \leq 2k \leq m} U_{2k}$$

$$\mathrm{SParHom}\,(V,W)\oplus\mathrm{SParHom}\,(W,V)=\bigoplus_{-m\le 2k+1\le m}U'_{2k+1}.$$

where

$$\begin{aligned} U_l &= \bigoplus_{i-j=l}\mathrm{ParHom}\,(E_j,E_i),\\ U'_l &= \bigoplus_{i-j=l}\mathrm{SParHom}\,(E_j,E_i). \end{aligned} \tag{3.18}$$

Therefore the deformation complex for $U(p,q)$-parabolic Higgs bundles can be written as

$$C^\bullet(E,\Phi):\ \bigoplus_{-m\le 2k\le m}U_{2k}\ \xrightarrow{\mathrm{ad}\,(\Phi)}\ \bigoplus_{-m\le 2k+1\le m}U'_{2k+1}\otimes K(D).$$

Each piece of this complex gives a subcomplex whose hypercohomology gives an eigenspace of the tangent space $T_{(E,\Phi)}\mathcal{U}$ for the circle action.

Proposition 3.7.6. *Let (E,Φ) be a stable $U(p,q)$-parabolic Higgs bundle which represents a fixed point of the circle action on $\mathcal{U}$. Then the eigenspace of the Hessian of f corresponding to the eigenvalue $-2k$ is $\mathbb{H}^1$ of the following complex*

$$C^\bullet(E,\Phi)_{2k}:U_{2k}\xrightarrow{\mathrm{ad}\,(\Phi)}U'_{2k+1}\otimes K(D)\,.$$

Proof. Let (E,Φ) be a stable parabolic $U(p,q)$ Higgs bundle. Then the circle action in the complex $C^\bullet(E,\Phi)$ is easily seen to be induced by the following map of deformation of complexes (for example, working at the level of cocycles)

$$\begin{array}{cccc} C^\bullet(E,\Phi): & \bigoplus U_{2k} & \xrightarrow{[-,\Phi]} & \bigoplus U_{2k+1}\otimes K(D)\\ \big\downarrow{\scriptstyle e^{\sqrt{-1}\theta}} & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle e^{\sqrt{-1}\theta}}\\ C^\bullet(E,e^{\sqrt{-1}\theta}\Phi): & \bigoplus U_{2k} & \xrightarrow{[-,e^{\sqrt{-1}\theta}\Phi]} & \bigoplus U_{2k+1}\otimes K(D). \end{array}$$

Now suppose that (E,Φ) is a critical point of f, i.e. $[(E,\Phi)]$ fixed point by the circle action on $\mathcal{U}$. Then we have a diagram like (3.7). So if v is an eigenvector of ψ_θ for the eigenvalue $e^{\sqrt{-1}l\theta}$, then

$$(\psi_\theta\otimes 1_{K(D)})\circ\Phi(v)=e^{\sqrt{-1}\theta}\Phi\circ\psi_\theta(v)=e^{\sqrt{-1}(l+1)\theta}\Phi(v).$$

So the isomorphism

$$\psi_\theta:(E,\Phi)\to(E,e^{\sqrt{-1}\theta}\Phi)$$

may be written with respect to the decomposition $E=\bigoplus E_l$ as the multiplication by $e^{\sqrt{-1}l\theta}$ on E_l. This ψ_θ induces an isomorphism between the deformation complexes $(C^\bullet(E,\Phi))$ and

$(C^\bullet(E, e^{\sqrt{-1}\theta}\Phi))$, by the adjoint $Ad(\psi_\theta)h = \psi_\theta h \psi_\theta^{-1}$. Since $Ad(\psi_\theta)$ is multiplication by $e^{\sqrt{-1}\theta l}$ on U_l, we can write the derivative of the action on the complexes, on each piece of degree l it is written as

$$\begin{array}{cccc} C^\bullet(E,\Phi)_{2l}: & U_{2l} & \xrightarrow{[-,\Phi]} & U_{2l+1}\otimes K(D) \\ \downarrow{\scriptstyle Ad(\psi_\theta)} & \downarrow{\scriptstyle e^{\sqrt{-1}2l\theta}} & & \downarrow{\scriptstyle e^{\sqrt{-1}(2l+1)\theta}} \\ C^\bullet(E,e^{\sqrt{-1}\theta}\Phi)_{2l}: & U_{2l} & \xrightarrow{[-,e^{\sqrt{-1}\theta}\Phi]} & U_{2l+1}\otimes K(D)\ . \end{array}$$

The circle action on $C^\bullet(E,\Phi)$ is given by the composition

$$C^\bullet(E,\Phi) \overset{e^{\sqrt{-1}\theta}}{\to} C^\bullet(E,e^{\sqrt{-1}\theta}\Phi) \overset{Ad(\psi_\theta)^{-1}}{\longrightarrow} C^\bullet(E,\Phi),$$

where its degree $2l$ piece is

$$\begin{array}{cccc} C^\bullet(E,\Phi)_{2l}: & U_{2l} & \xrightarrow{[-,\Phi]} & U_{2l+1}\otimes K(D) \\ \downarrow & \downarrow{\scriptstyle e^{-\sqrt{-1}2l\theta}} & & \downarrow{\scriptstyle e^{-\sqrt{-1}2l\theta}} \\ C^\bullet(E,\Phi)_{2l}: & U_{2l} & \xrightarrow{[-,\Phi]} & U_{2l+1}\otimes K(D)\ . \end{array}$$

So the action on $\mathbb{H}^1(C^\bullet(E,\Phi))$ is simply multiplication by $e^{-\sqrt{-1}2l\theta}$ on $\mathbb{H}^1(C^\bullet(E,\Phi)_{2l})$. This completes the proof. $\square$

Corollary 3.7.7. *(E,Φ) is a local minimum of f if and only if $\mathbb{H}^1(C^\bullet(E,\Phi)_{2k}) = 0$ for all $k \geq 1$.*

Proposition 3.7.8. *Let (E,Φ) be a stable $U(p,q)$-parabolic Higgs bundle which is a fixed point of the S^1-action on $\mathcal{U}$. Then $\chi(C^\bullet(E,\Phi)_{2k}) \leq 0$ for all $k \geq 1$, and equality holds if and only if*

$$\mathrm{ad}\,(\Phi) : U_{2k} \to U'_{2k+1}\otimes K(D)$$

is an isomorphism of bundles.

Proof. We want to get a bound for

$$\chi(C^\bullet(E,\Phi)_{2k}) = \chi(U_{2k}) - \chi(U'_{2k+1}\otimes K(D)).$$

We denote $\Phi_{2k} = \mathrm{ad}\,(\Phi)|_{U_{2k}} : U_{2k} \to U'_{2k+1}\otimes K(D)$. We also introduce the maps $\Phi_{2k+1} = \mathrm{ad}\,(\Phi) : U_{2k+1} \to U'_{2k+2}\otimes K(D)$, by using the definition (3.18).

The dual of each U_l is

$$U_l^* = \bigoplus_{i-j=l}(\mathrm{ParHom}\,(E_j,E_i))^* = \bigoplus_{i-j=l}\mathrm{SParHom}\,(E_i,E_j(D)) = U'_{-l}(D).$$

The dual of Φ_{2k} is

$$\Phi_{2k}^t = \Phi_{-2k-1} \otimes 1_{K^{-1}} : U_{-2k-1} \otimes K^{-1} \to U'_{-2k}(D).$$

Using the Hitchin-Kobayashi correspondence for parabolic Higgs bundles one can show that the stability of (E, Φ) implies the polystability of $(\mathrm{ParEnd}\,(E), \mathrm{ad}\,(\Phi))$ as a parabolic Higgs bundle, where $\mathrm{ad}\,(\Phi) : \mathrm{ParEnd}\,(E) \to \mathrm{SParEnd}\,(E) \otimes K(D)$.

The parabolic structure on E induces a parabolic structure on $\mathrm{ParEnd}\,(E)$. With this parabolic structure $(\mathrm{ParEnd}\,(E), \mathrm{ad}\,(\Phi))$ is a parabolic Higgs bundle. Now, the stability of $(E\,\Phi)$ implies the polystability (in particular semistability) of $(\mathrm{ParEnd}\,(E), \mathrm{ad}\,(\Phi))$. This can be seen, for example, producing a solution to the Hitchin equations on $(\mathrm{ParEnd}\,(E), \mathrm{ad}\,(\Phi))$ out of the solutions on (E, Φ). The degree of any subbundle of a parabolic bundle is smaller than the parabolic degree, and the parabolic degree of $\mathrm{ParEnd}\,(E)$ is zero. Hence, since $\ker(\Phi_{2k})$ and $\ker(\Phi_{-2k-1})$ are $\mathrm{ad}\,(\Phi)$-invariant subbundles of $(\mathrm{ParEnd}\,(E), \mathrm{ad}\,(\Phi))$, we have $\deg(\ker(\Phi_{2k})) \leq 0$ and $\deg(\ker(\Phi_{-2k-1})) \leq 0$. Therefore we have the following chain of inequalities

$$\begin{aligned}
\deg(U_{2k}) &= \deg(\ker(\Phi_{2k})) + \deg(\mathrm{Im}\,(\Phi_{2k})) \leq \deg(\mathrm{Im}\,(\Phi_{2k})) \leq -\deg(\mathrm{Im}\,(\Phi_{2k}^t)) \\
&= -\deg(\mathrm{Im}\,(\Phi_{-2k-1} \otimes 1_{K^{-1}})) = -\deg(\mathrm{Im}\,(\Phi_{-2k-1})) + \mathrm{rk}\,(\mathrm{Im}\,(\Phi_{-2k-1}))(2g-2) \\
&= \deg(\ker(\Phi_{-2k-1})) - \deg(U_{-2k-1}) + \mathrm{rk}\,(\mathrm{Im}\,(\Phi_{-2k-1}))(2g-2) \\
&\leq -\deg(U_{-2k-1}) + \mathrm{rk}\,(\mathrm{Im}\,(\Phi_{-2k-1}))(2g-2) \\
&= \deg(U'_{2k+1}(D)) + \mathrm{rk}\,(\mathrm{Im}\,(\Phi_{2k}))(2g-2),
\end{aligned}$$

where we have used that $\mathrm{rk}\,(\mathrm{Im}\,(h)) = \mathrm{rk}\,(\mathrm{Im}\,(h^t))$ and that $\deg(\mathrm{Im}\,(h)) \leq -\deg(\mathrm{Im}\,(h^t))$ for any morphism of sheaves h.

Finally, we use this result in the computation of the Euler characteristic.

$$\begin{aligned}
\chi(C^\bullet(E, \Phi)_{2k}) &= \deg(U_{2k}) + \mathrm{rk}\,(U_{2k})(1-g) - \deg(U_{2k+1} \otimes K(D)) - \mathrm{rk}\,(U_{2k+1})(1-g) \\
&= \deg(U_{2k}) + \mathrm{rk}\,(U_{2k})(1-g) - \deg(U_{2k+1}) - \mathrm{rk}\,(U_{2k+1})(g-1+s) \\
&\leq \deg(U'_{2k+1}(D)) + \mathrm{rk}\,(\mathrm{Im}\,(\Phi_{2k}))(2g-2) + \mathrm{rk}\,(U_{2k})(1-g) - \deg(U_{2k+1}) \\
&\quad - \mathrm{rk}\,(U_{2k+1})(g-1+s) \\
&= (g-1)(2\,\mathrm{rk}\,(\mathrm{Im}\,(\Phi_{2k})) - \mathrm{rk}\,(U_{2k}) - \mathrm{rk}\,(U_{2k+1})),
\end{aligned}$$

where we have used that $U'_{2k+1} = U_{2k+1}$ since all the weights are different and of multiplicity 1, and hence for $i \neq j$ it is $\mathrm{SParHom}\,(E_i, E_j) = \mathrm{ParHom}\,(E_i, E_j)$, since E_i and E_j are different pieces in the decomposition of Proposition 3.7.4.

So $\chi(C^\bullet(E,\Phi)_{2k}) \leq 0$. If equality holds then $\mathrm{rk}\,(\Phi_{2k}) = \mathrm{rk}\,(U_{2k}) = \mathrm{rk}\,(U_{2k+1})$ and $\mathrm{ad}\,(\Phi)|_{U_{2k}}$ is generically an isomorphism of sheaves. In this case $\chi(C^\bullet(E,\Phi)_{2k}) = 0$ if and only if $\mathrm{ad}\,(\Phi)|_{U_{2k}}$ is an isomorphism of bundles. □

Corollary 3.7.9. *Let (E,Φ) be a stable $U(p,q)$-parabolic Higgs bundle which represents a critical point of the Morse function f. This critical point is a minimum if and only if*

$$\mathrm{ad}\,(\Phi)|_{U_{2k}} : U_{2k} \to U'_{2k+1} \otimes K(D)$$

is an isomorphism for all $k \geq 1$.

Proof. By Corollary 3.7.7, (E,Φ) is a local minimum if and only if

$$\mathbb{H}^1(C^\bullet(E,\Phi)_{2k}) = 0, \quad \forall k \geq 1. \tag{3.19}$$

Now, by Proposition 3.2.2, $\mathbb{H}^0(C^\bullet(E,\Phi)_{2k}) = 0$ and $\mathbb{H}^2(C^\bullet(E,\Phi)_{2k}) = 0$, for $k \geq 1$, hence (E,Φ) is a local minimum if and only if

$$\chi(C^\bullet(E,\Phi)_{2k}) = \sum(-1)^i \dim \mathbb{H}^i(C^\bullet(E,\Phi)_{2k})) = 0, \quad \forall k \geq 1.$$

But, by Proposition 3.7.8, this is equivalent to ask that

$$\mathrm{ad}\,(\Phi) : U_{2k} \to U'_{2k+1} \otimes K(D)$$

be an isomorphism of sheaves. □

Finally, we show that all these minima are in $\mathcal{N}$.

Proposition 3.7.10. *Let $(E,\Phi) = (E_0 \oplus \cdots \oplus E_m, \Phi)$ be a fixed point of the circle action, with $m \geq 2$. Then (E,Φ) is not a local minimum.*

Proof. If m is even then

$$\Phi_m : \mathrm{ParHom}\,(E_0, E_m) \to \mathrm{SParHom}\,(E_0, 0) \otimes K(D)$$

cannot be generically an isomorphism, since $\mathrm{ParHom}\,(E_0, E_m) \neq 0$ and $\mathrm{SParHom}\,(E_0, 0) = 0$. By corollary 3.7.9, (E,Φ) is not a local minimum.

If $m \geq 2$ is odd, consider the homomorphism

$$\Phi_{m-1} : \mathrm{ParHom}\,(E_0, E_{m-1}) \oplus \mathrm{ParHom}\,(E_1, E_m) \to \mathrm{SParHom}\,(E_0, E_m) \otimes K(D).$$

We will show that Φ_{m-1} is not injective, and therefore $(E_0 \oplus \cdots \oplus E_m, \Phi)$ is not a minimum. We need to find $\zeta = (\zeta_1, \zeta_2) \in U_{m-2}$, $\zeta \neq 0$ such that $\Phi_{m-2}(\zeta) = 0$, i.e. we need to find ζ_1 and ζ_2 making the following diagram commutative.

$$\begin{array}{ccc} E_0 & \xrightarrow{\Phi} & E_1 \otimes K(D) \\ \Big\downarrow{\scriptstyle \zeta_1} & & \Big\downarrow{\scriptstyle \zeta_2 \otimes 1_{K(D)}} \\ E_{m-1} & \xrightarrow{\Phi} & E_m \otimes K(D) \end{array}$$

For this, take $\zeta_2 \neq 0$ such that $\zeta_2 \otimes 1_{K(D)}(E_1 \otimes K(D)) \subset \Phi(E_{m-1})$ and take ζ_1 such that

$$\Phi \circ \zeta_1 = (\zeta_2 \otimes 1_{K(D)}) \circ \Phi,$$

therefore $\Phi_{m-1}(\zeta) = (\zeta_2 \otimes 1_{K(D)}) \circ \Phi - \Phi \circ \zeta_1 = 0$ with $\zeta \neq 0$. So Φ_{m-1} is not injective. $\square$

Theorem 3.7.11. *The set of local minima of $f : \mathcal{U}(p, q, a, b; \alpha, \eta) \to \mathbb{R}$ coincides with $\mathcal{N}(p, q, a, b; \alpha, \eta)$.*

Proof. By Proposition 3.7.10, for (E, Φ) to be a minimum (E, Φ) must have a decomposition of the form $E = E_0 \oplus E_1$ with Φ mapping E_0 into $E_1 \otimes K(D)$. But by assumption the only possible decompositions are $E = V \oplus W$ with $\Phi = \gamma$ and $E = W \oplus V$ with $\Phi = \beta$. So $(E, \Phi) \in \mathcal{N}$.

Conversely, if $(E, \Phi) \in \mathcal{N}$ then $m = 1$ and $U_{2k} = U'_{2k+1} = 0$, for $k \geq 1$. So Corollary 3.7.9 applies and (E, Φ) is a minimum. $\square$

Which of the two components of the Higgs field vanishes is given by the following.

Lemma 3.7.12. *Let $(E, \Phi) \in \mathcal{N}$. Then the Toledo invariant $\tau \neq 0$ and*

(i) $\gamma = 0$ if and only if $\tau < 0$.

(ii) $\beta = 0$ if and only if $\tau > 0$.

Proof. Observe that τ can not be equal to zero because this implies $\gamma = \beta = 0$ and then (E, Φ) cannot be stable. The rest follows directly from the definition of the Toledo invariant. $\square$

3.8 Parabolic triples and parabolic $\mathrm{U}(p,q)$-Higgs bundles.

Let (E,Φ) be a $\mathrm{U}(p,q)$-parabolic Higgs bundle with $\Phi=\beta : W \to V\otimes K(D)$. This defines a parabolic triple $T=(E_1,E_2,\phi)$ as in Chapter 2 where $E_1=V\otimes K$, $E_2=W$, $\phi=\beta$. Conversely, given a parabolic triple $T=(E_1,E_2,\phi)$ we get a $U(p,q)$-parabolic Higgs bundle with $\Phi=\beta$ by defining $(E=V\oplus W,\Phi)$ where $V=E_1\otimes K^{-1}$, $W=E_2$ and $\beta=\phi$. When (E,Φ) is a $U(p,q)$-parabolic Higgs bundle with $\Phi=\gamma : V\to W\otimes K(D)$ we have an analogous correspondence. That is, the corresponding triple to (E,Φ) is $T=(W\otimes K,V,\gamma)$.

Lemma 3.8.1. *A $\mathrm{U}(p,q)$-parabolic Higgs bundle (E,Φ) with $\beta=0$ or $\gamma=0$ is parabolically stable if and only if the corresponding parabolic triple is σ-stable for $\sigma=2g-2$.*

Proof. Let $T=(E_1,E_2,\phi)$ be the triple defined by (E,Φ) (without loss of generality we assume $\gamma=0$). Therefore if we set $\sigma=2g-2$ we have

$$\begin{aligned}\mathrm{par}\mu_\sigma(T) &= \frac{\mathrm{pardeg}\,(E_1)+\mathrm{pardeg}\,(E_2)}{\mathrm{rk}\,(E_1)+\mathrm{rk}\,(E_2)}+\sigma\frac{\mathrm{rk}\,(E_2)}{\mathrm{rk}\,(E_1)+\mathrm{rk}\,(E_2)}\\ &= \frac{\mathrm{pardeg}\,(V)+\mathrm{pardeg}\,(W)+p(2g-2)}{p+q}+\sigma\frac{q}{p+q} \\ &= \mathrm{par}\mu\,(E)+2g-2.\end{aligned} \tag{3.20}$$

The correspondence between parabolic triples and $\mathrm{U}(p,q)$ parabolic bundles with $\beta=0$ or $\gamma=0$ gives also a correspondence between parabolic subtriples and parabolic subbundles. That is, given a subtriple T' of T the corresponding $\mathrm{U}(p,q)$ parabolic Higgs bundle is a Φ-invariant subbundle of (E,Φ), and conversely given (E',Φ') the corresponding triple gives a parabolic subtriple of T. Hence equation (3.20) gives that $\mathrm{par}\mu_{2g-2}(T')<\mathrm{par}\mu_{2g-2}(T)$ if and only if $\mathrm{par}\mu\,(E')<\mathrm{par}\mu\,(E)$. □

Remark 3.8.2. The genericity condition on the weights imposes that there are no properly semistable $\mathrm{U}(p,q)$-parabolic Higgs bundles. Therefore there are no properly σ-semistable parabolic triples. This means that $\sigma=2g-2$ is not a critical value in the sense of Section 2.3

Combining Lemma 3.7.12 and Lemma 3.8.1, we have the following.

Proposition 3.8.3. *Let $\mathcal{N}(p,q,a,b,\alpha,\eta)$ be the submanifold of local minima of $\mathcal{U}(p,q,a,b,\alpha,\eta)$ and let τ be the Toledo invariant then,* *(i) If $\tau < 0$ then $\mathcal{N}(p,q,a,b,\alpha,\eta) = \mathcal{N}_{2g-2}(p,q,a+p(2g-2),b,\alpha,\eta)$.*

(ii) If $\tau > 0$ then $\mathcal{N}(p,q,a,b,\alpha,\eta) = \mathcal{N}_{2g-2}(q,p,b+q(2g-2),a,\eta,\alpha)$.

□

It is then clear that in order for $\mathcal{N}(p,q,a,b,\alpha,\eta)$ to be non-empty, $2g-2$ must be in the range for σ given by Proposition 2.1.9, where σ_m and σ_M are determined by the correspondence given in Proposition 3.8.3. In fact, one has the following.

Proposition 3.8.4. *Let $\sigma_m = \sigma_m(p,q,a,b;\alpha,\eta)$ and $\sigma_M = \sigma_M(p,q,a,b;\alpha,\eta$ be the bounds for σ defined in Proposition 2.1.9 for the moduli space of parabolic triples identified in Proposition 3.8.3 with the subvariety $\mathcal{N}(p,q,a,b;\alpha,\eta)$. Then*

$$0 < |\tau| < \min\{p,q\}(2g-2+s) \Leftrightarrow \begin{cases} \sigma_m(p,q,a,b) < 2g-2 < \sigma_M(p,q,a,b) & \text{if } p \neq q \\ \sigma_m(p,q,a,b) < 2g-2 & \text{if } p = q \end{cases} \tag{3.21}$$

Proof. Write σ_m and σ_M in terms of τ, that is,

$$\begin{cases} \sigma_m(p,q,a,b) = \frac{(p+q)}{2pq}\tau + 2g-2 & \text{if } \tau < 0 \\ \sigma_m(p,q,a,b) = -\frac{p+q)}{2pq}\tau + 2g-2 & \text{if } \tau > 0 \\ \sigma_M(p,q,a,b) = (1+\frac{p+q}{|p-q|})(\frac{(p+q)}{2pq}\tau + 2g-2) + s\frac{p+q}{|p-q|} & \text{if } \tau < 0 \\ \sigma_M(p,q,a,b) = (1+\frac{p+q}{|p-q|})(-\frac{(p+q)}{2pq}\tau + 2g-2) + s\frac{p+q}{|p-q|} & \text{if } \tau > 0 \end{cases} \tag{3.22}$$

and use the Milnor-Wood inequality for the parabolic Toledo invariant τ given in corollary 3.4.3

□

Remark 3.8.5. This last proposition refine the condition that we note in Remark 3.3.3 for the number of marked points in order to the existence of $\mathcal{N}$ when the genus of the curve X has low genus. Observe that $\mathcal{N}$ non empty implies the following:

(i) If $g = 0$ then $s \geq 3$,

(ii) If $g = 1$ then $s \geq 1$,

and for genus $g \geq 2$ there is no condition on the number of marked points s.

3.9 Number of connected components

Proposition 3.9.1. *Let σ_L be the maximal critical value given in (2.20). Let $\mathcal{N}(n,1,a,b;\alpha,\eta)$ be the submanifold of local minima of $\mathcal{U}(n,1,a,b;\alpha,\eta)$. Let $\tau_L = 2g-2+\frac{A}{n}$ the maximal value of the parabolic Toledo invariant. Then,*

$$0 < |\tau| < \tau_L \Leftrightarrow \sigma_m(n,1,a,b) < 2g-2 < \sigma_L(n,1,a,b) \tag{3.23}$$

Proof. In the case $p = n$ and $q = 1$ we can use the maximal value of σ, that is σ_L given in (2.20). Hence, repeating the same arguments as in Proposition 3.8.4 for σ_L we obtain

$$\tau_L = 2g - 2 + \frac{A}{n} \tag{3.24}$$

where A is defined as in Proposition 2.7.9.

□

Theorem 3.9.2. *The subvariety $\mathcal{N}(n,1,a,b;\alpha,\eta)$ is non-empty and connected if and only if the parabolic Toledo invariant satisfies the bound given in Propositin 3.9.1, i.e. $|\tau| < \tau_L$.*

Proof. The result will follow from Proposition 3.8.3 and Corollary 2.8.4. □

Combining Theorem 3.9.2 and Corollaries 3.7.11 and 3.7.2 we obtain one of the main results of our thesis.

Theorem 3.9.3. *Let X be a compact Riemann surface of genus $g \geq 1$, and let α and η be generic weights. Then, the moduli space $\mathcal{U}(p,q,a,b;\alpha,\eta)$ of parabolic $\mathcal{U}(n,1)$-Higgs bundles with full flags is non-empty and connected if and only if $|\tau| < \tau_L$.*

Particularize Theorem 3.6.2 to the case $p = n$ and $q = 1$, we obtain the following.

Theorem 3.9.4. *Let (n,a,α) and $(1,b,\eta)$ be such that* $\mathrm{pardeg}\,(V) + \mathrm{pardeg}\,(W) = 0$. *Then there is a homeomorphism*

$$\mathcal{R}(n,1;\alpha,\eta) \cong \bigsqcup_{a,b} \mathcal{U}(n,1,a,b;\alpha,\eta).$$

Note that $(n,1,a,b,\alpha,\eta)$ must also satisfy the Milnor–Wood inequality, which in this cases reduces to

$$\mathrm{pardeg}\,(V) \leq (g - 1 + s/2).$$

Combining Theorems 3.9.4 and 3.9.3 we have the following.

Theorem 3.9.5. *Let X be a compact Riemann surface and $S = \{x_i\}_{i=1}^{s}$ a finite set of marked points on X. Let $\mathcal{R}(n, 1; \alpha, \eta)$ be the moduli space of representations of $\pi_1(X - S)$ in $\mathrm{U}(n, 1)$ with fixed holonomy classes in $\mathrm{U}(n) \times \mathrm{U}(1)$. For α and η generic weights, the number of non-empty connected components of $\mathcal{R}(n, 1; \alpha, \eta)$ equals the number of integers a such that*

$$a + \sum_{x \in S} (\tilde{\alpha}_1(x) + \ldots + \tilde{\alpha}_p(x)) \leq \tau_L/2,$$

where τ_L is given by (3.24).

Chapter 4

Betti numbers of the moduli space of parabolic U(2, 1)-Higgs bundles.

4.1 Bott–Morse theory for the moduli space of parabolic U(2, 1)-Higgs bundles.

It is known from [GGM] that for fixed rank n, and different choices of degrees and generic weights, the moduli spaces of stable parabolic $GL(n, \mathbb{C})$-Higgs bundles have the same Betti numbers. For the moduli of stable parabolic U(2, 1)-Higgs bundles, we shall show that the Betti numbers depend not only on the degrees but also on the weight type of the parabolic bundles. Recall from Section 3.1 that the moduli of stable parabolic U(2, 1)-Higgs bundles is a closed subvariety of the moduli of stable parabolic $GL(3, \mathbb{C})$-Higgs bundles. Therefore, the diffeomorphism that takes a moduli space of parabolic $GL(n, \mathbb{C})$-Higgs bundles with fixed weights to another one corresponding to a different system of weights does not interchange their respective subvarieties of stable parabolic U(2, 1)-Higgs bundle.

Let $\mathcal{U}(2, 1, a, b; \alpha, \eta)$ be the moduli space of parabolic U(2, 1)-Higgs bundles with ranks $\operatorname{rk}(V) = 2$, $\operatorname{rk}(W) = 1$, degrees $\deg(V) = a$, $\deg(W) = b$ and systems of weights α and η on V and W respectively. When it does not induce confusion we denote the moduli of stable parabolic U(2, 1)-Higgs bundles by $\mathcal{U} = \mathcal{U}(2, 1, a, b; \alpha, \eta)$ and the moduli of stable parabolic $GL(3, \mathbb{C})$-Higgs bundles by $\mathcal{M} = \mathcal{M}(3, a + b; \alpha \cup \eta)$.

Proposition 4.1.1. *Let L be a parabolic line bundle of degree l and system of weights δ, then the map $V \oplus W \mapsto (V \oplus W) \otimes L$ induces an isomorphism from the moduli space*

$\mathcal{U}(p, q, a, b, \alpha, \eta)$ *to the moduli space* $\mathcal{U}(p, q, a', b', \alpha', \eta')$ *where* a',b',α',η' *are given following the definition of the parabolic tensor product (see 1.1.11).*

The map $V \oplus W \to V^* \oplus W^*$ *induces an isomorphism from the moduli space* $\mathcal{U}(p, q, a, b, \alpha, \eta)$ *to* $\mathcal{U}(p, q, a', b', \alpha', \eta')$*, where* $a' = ps - a$, $b' = qs - b$*, and the system of weights are given by the notion of parabolic dual (see 1.1.5).*

The Toledo invariant for the moduli of parabolic $\mathrm{U}(p, q)$-Higgs bundles was defined in 3.4.1. For $(E, \Phi) \in \mathcal{U}(2, 1, a, b; \alpha, \eta)$, it is

$$\tau = \frac{2}{3}\left(\Delta - 3b + \sum_{x \in D}(\alpha_1(x) + \alpha_2(x) - \eta(x))\right). \tag{4.1}$$

Where $\Delta = a + b$.

Remark 4.1.2. Having generic weights implies that τ can not be zero.

Lemma 4.1.3. *Assume that there is at least one point where this weights are different. The moduli space* $\mathcal{U}(2, 1, a, b; \alpha, \eta)$*, with any* $\tau \in \mathbb{R}$ *and* $\Delta \in \mathbb{Z}$*, is isomorphic to another moduli space* $\mathcal{U}(2, 1, a', b'; \alpha', \eta')$ *with* $\tau' = -\tau < 0$ *and* $\Delta' \equiv 0(3)$*.*

Proof. If $\tau > 0$ then Proposition 4.1.1 (second part) says that $\mathcal{U}(2, 1, a, b; \alpha, \eta)$ is isomorphic to $\mathcal{U}(2, 1, 2s - a, s - b; \alpha', \eta')$. Now

$$\tau' = \frac{2\,\mathrm{pardeg}\,(W^*) - \mathrm{pardeg}\,(V^*)}{3} = \frac{-2\,\mathrm{pardeg}\,(W) + \mathrm{pardeg}\,(V)}{3} = -\tau.$$

Now assume $\tau \leq 0$. Then if Δ is not equivalent to $0(3)$, take a trivial line bundle where all the weights are zero and one weight is $\delta(x) \in (0, 1)$, $x \in D$. This gives a parabolic line bundle L. Using Proposition 4.1.1 (first part), $\mathcal{U}(2, 1, a, b; \alpha, \eta)$ is isomorphic to some $\mathcal{U}(2, 1, a', b'; \alpha', \eta')$. First $\tau' = \tau$. Second choosing δ such that $\alpha_1(x) + \delta(x)$, $\alpha_2(x) + \delta(x)$, $\eta(x) + \delta(x)$ exactly one of them (resp. two) is bigger than 1, then $\Delta' = a' + b' = a + 1 + b = \Delta + 1$ (resp. $+2$). ◻

Consider the action of S^1 on $\mathcal{U}$ given in Section 3.7. Recall that this action provides us with a Hamiltonian action on the set of solutions to Hitchin's equations $\mathcal{S}$. With respect to one of the complex structures, this is a Hamiltonian action and the associated moment map is a proper map on $\mathcal{U}$,

$$f : \mathcal{U} \to \mathbb{R}$$
$$f([E, \Phi]) = \|\Phi\|^2.$$

Recall also from Section 3.7 that, by a result of Frankel, we get that f is also a perfect Bott-Morse function. Since f is perfect, Bott-Morse theory gives the following formula for the Poincaré polynomial of a manifold,

$$P_t(\mathcal{U}) = \sum_{\mathcal{N}} t^{\lambda_{\mathcal{N}}} P_t(\mathcal{N}), \tag{4.2}$$

where the sum runs over all critical submanifolds $\mathcal{N}$ of $\mathcal{U}$ for f and $\lambda_{\mathcal{N}}$ is the Morse index of f on $\mathcal{N}$.

Hence to compute the Poincaré polynomial of $\mathcal{U}$ we have to compute the indices and the Poincaré polynomials of each critical submanifold of $\mathcal{U}$ for f.

Note that for the computation of the number of connected components of the moduli space of parabolic $U(p,q)$-Higgs bundles we only considered the minima. Here, we need to study all the critical submanifolds.

From Proposition 3.7.4 we get a description of the critical submanifolds of $\mathcal{U}$. They correspond to parabolic Higgs bundles which decompose in either of these three ways

$$\begin{array}{llll}
(E = E_0 \oplus E_1, \Phi), & \mathrm{rk}\,(E_0) = 1, & \mathrm{rk}\,(E_1) = 2, & \Phi : E_0 \to E_1 \otimes K(D). \\
(E = E_0 \oplus E_1, \Phi), & \mathrm{rk}\,(E_0) = 2, & \mathrm{rk}\,(E_1) = 1, & \Phi : E_0 \to E_1 \otimes K(D). \\
(E = E_0 \oplus E_1 \oplus E_2, \Phi), & \mathrm{rk}\,(E_i) = 1, & i = 1,2,3, & \Phi : E_0 \to E_1 \otimes K(D), \\
 & & & \Phi : E_1 \to E_2 \otimes K(D).
\end{array}$$

This classifies the critical submanifolds according to three categories. We shall say that a critical submanifold is of type $(1,2)$, $(2,1)$, or $(1,1,1)$ respectively depending on what type of splitting corresponds to. We denote by $\mathcal{N}(1,2)$, $\mathcal{N}(2,1)$, and $\mathcal{N}(1,1,1)$ the union of all critical submanifolds of type $(1,2)$, $(2,1)$, and $(1,1,1)$ respectively.

Observe that the critical submanifolds of type $(1,2)$ and type $(2,1)$ consist of parabolic Higgs bundles for which either $\gamma = 0$ or $\beta = 0$, respectively. These, as we computed in Section 3.7, are the minima of the function f. These critical submanifolds are identified with the moduli spaces of stable parabolic triples $\mathcal{N}_{2g-2}(2,1,d_0,d_1;\eta,\alpha)$, and $\mathcal{N}_{2g-2}(1,2,d_1,d_0;\alpha,\eta)$ by Lemma 3.8.1.

In the case of critical submanifolds of type $(1,1,1)$, the Higgs bundle decomposes as $E = E_0 \oplus E_1 \oplus E_2$. So we will be dealing with parabolic chains, which by definition are n-tuples of parabolic bundles together with $(n-1)$-tuples of strongly parabolic homomorphisms $\phi_i : E_i \to E_{i-1}(D)$. This is the reason for restricting attention to $p = 2$ and $q = 1$. To compute the Betti numbers for other values of p and q we have to deal with parabolic chains involving parabolic bundles of rank bigger than one, and this tool has not been developed yet. This is left for future work.

During the following sections we shall compute the Poincaré polynomials that take part in the formula (4.2). By the discussion above,

$$P_t(\mathcal{U}) = \begin{cases} P_t(\mathcal{U};(2,1)) + P_t(\mathcal{U};(1,1,1)) & \text{for } \tau > 0 \\ P_t(\mathcal{U};(1,2)) + P_t(\mathcal{U};(1,1,1)) & \text{for } \tau < 0 \end{cases} \tag{4.3}$$

where we denote

$$P_t(\mathcal{U};(2,1)) = P_t\mathcal{N}(2,1),$$
$$P_t(\mathcal{U};(1,2)) = P_t\mathcal{N}(1,2)$$

and

$$P_t(\mathcal{U};(1,1,1)) = \sum_{\mathcal{N} \text{ of type } (1,1,1)} t^{\lambda_{\mathcal{N}}} P_t(\mathcal{N}).$$

Through these sections our computations will depend on the degrees and weights. However by Lemma 4.1.3 the Toledo invariant τ and Δ can be arrange to satisfy $\tau < 0$ and $\Delta \equiv 0(3)$, at least when the three weights at one point $x \in D$ are distinct.

Recall from Theorem 3.9.2.

Theorem 4.1.4. *The moduli space $\mathcal{U}(2,1,a,b;\alpha,\eta)$ is non-empty and connected if and only if the parabolic Toledo invariant satisfies the bound given in Propositin 3.9.1, i.e. $|\tau| < \tau_L$.*

4.2 Critical subvarieties of type (1,1,1).

We start with the case where $(E = V \oplus W, \Phi)$ splits as a direct sum of three line bundles $E = E_0 \oplus E_1 \oplus E_2$ where E_0 and E_2 are contained in V, together with strongly parabolic homomorphisms $\Phi_0 = \gamma|_{E_0} : E_0 \to E_1 \otimes K(D)$ and $\Phi_1 = \beta : E_1 \to E_2 \otimes K(D)$.

We denote along this section $d_i = \deg(E_i)$ so $\Delta = d_0 + d_1 + d_2 = d_0 + b + d_2$ i.e.

$$\begin{aligned} a &= d_0 + d_2 \\ b &= d_1. \end{aligned} \tag{4.4}$$

Each distribution of the weights for E_0 and E_2 is given by a collection of injective maps $\varpi = \{\varpi_x : \{1,2\} \to \{1,2\};\ x \in D\}$ such that the weight of E_0 at $x \in D$ is $\alpha_{\varpi_x(1)}(x)$ and the weight of E_2 at $x \in D$ is $\alpha_{\varpi_x(2)}(x)$.

Let $\mathcal{N}$ be one critical submanifold of type $(1,1,1)$. Then it has an associated degree d_0 satisfying (4.4) and a distribution of weights ϖ. Several critical submanifolds (or none) may have a given d_0 and ϖ.

Proposition 4.2.1. *Let $\mathcal{N}$ be a critical submanifold of type (1,1,1), the Morse index of $\mathcal{N}$ depends only on d_0 and ϖ, and it is given by*

$$\lambda_{\mathcal{N}} = 2g - 2 + 2(2d_0 - \Delta + b) + 2(s - v)), \tag{4.5}$$

where $v = \sharp\{x \in D;\ \alpha_{\varpi_x(1)}(x) \leq \alpha_{\varpi_x(2)}(x)\}$, and s is the number of marked points.

Proof. By proposition 3.7.6 the Morse index equals the dimension of $\mathbb{H}^1(C_1^\bullet)$ where $C_1^\bullet$ is, in our situation, the complex

$$\mathrm{ParHom}\,(E_0, E_2) \to 0.$$

Using the long exact sequence for this complex we get $\mathbb{H}^0(C_1^\bullet) = 0$ since it is isomorphic to $H^0(\mathrm{ParHom}\,(E_0, E_2))$ and this is zero since $\deg(\mathrm{ParHom}\,(E_0, E_2)) < 0$. Hence,

$$\begin{aligned}
\frac{1}{2}\lambda_{\mathcal{N}} &= \dim T_E\mathcal{U}_{<0} = \dim \mathbb{H}^1(C_1^\bullet)\\
&= \dim H^1(\mathrm{ParHom}\,(E_0, E_2)) = -\chi(\mathrm{ParHom}\,(E_0, E_2))\\
&= -\deg(\mathrm{ParHom}\,(E_0, E_2)) - \mathrm{rk}\,(\mathrm{ParHom}\,(E_0, E_2))(1-g)\\
&= d_0 - d_2 + s - \sum_{x\in D} \dim \mathrm{ParHom}\,(E_{0,x}, E_{2,x}) + g - 1.
\end{aligned}$$

Hence, $\lambda_{\mathcal{N}} = 2g - 2 + 2(2d_0 + b - \Delta) + 2(s - v)$, where $v = \sharp\{x \in D; \alpha_{\varpi_x(1)}(x) \leq \alpha_{\varpi_x(2)}(x)\}$. ∎

Remark 4.2.2. Let $\mathcal{N}(d_0, \varpi)$ be the union of all critical submanifolds corresponding to degree d_0 and distribution of weights ϖ. Then, we may decompose

$$\mathcal{N}(1,1,1) = \bigsqcup_{d_0,\varpi} \mathcal{N}(d_0, \varpi).$$

From now on we denote

$$v_1 = \sharp\{x \in D;\ \alpha_{\varpi(1)}(x) < \eta(x)\},$$

$$v_2 = \sharp\{x \in D;\ \eta(x) < \alpha_{\varpi(2)}(x)\}.$$

Proposition 4.2.3. *Assume $\tau < 0$. Fix d_0 and ϖ. Then there is an isomorphism*

$$\begin{aligned} \mathcal{N}(d_0, \varpi) &\longrightarrow \operatorname{Jac}^{d_0} X \times S^{m_1} X \times S^{m_2} X \\ (E_0 \oplus E_1 \oplus E_2, \Phi_0, \Phi_1) &\mapsto (E_0, \operatorname{div}(\Phi_0), \operatorname{div}(\Phi_1)), \end{aligned}$$

where

$$\begin{aligned} m_1 &= \deg(\operatorname{SParHom}(E_0, E_1) \otimes K(D)) = b - d_0 + 2g - 2 + v_1 \\ m_2 &= \deg(\operatorname{SParHom}(E_1, E_2) \otimes K(D)) = \Delta - d_0 - 2b + 2g - 2 + v_2 \end{aligned}$$

Furthermore, the degree d_0 of E_0 is bounded below by $\bar{d}_0$, that is,

$$d_0 \geq \bar{d}_0 = \left[\frac{1}{3} \left(\Delta + \sum_{x \in D} (\eta(x) + \alpha_{\varpi_x(2)}(x) - 2\alpha_{\varpi_x(1)}(x)) \right) + 1 \right], \tag{4.6}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. There is exactly one component for fixed d_0 and ϖ for $m_1 \geq 0$, $m_2 \geq 0$ and $d_0 \geq \bar{d}_0$. Conversely, if $d_0 \geq \bar{d}_0$ then $\mathcal{N}(d_0, \varpi)$ is non-empty.

Proof. The isomorphism is obvious (see [GGM]). The stability condition for (E, Φ) applied to the subbundles E_2 and $E_1 \oplus E_2$, together with the formula $d_2 = \Delta - b - d_0$, gives the following two bounds for d_0:

$$2\Delta - 3b - \sum_{x \in D} (\alpha_{\varpi_x(1)}(x) + \eta(x) - 2\alpha_{\varpi_x(2)}(x)) < 3d_0, \tag{4.7}$$

$$\Delta - \sum_{x \in D} (2\alpha_{\varpi_x(1)}(x) - \eta(x) - \alpha_{\varpi_x(2)}(x)) < 3d_0. \tag{4.8}$$

To determine which is a sharper bound we subtract the left hand sides of these two inequalities. This gives a positive multiple of τ, hence for $\tau > 0$ the first inequality is sharper, for $\tau < 0$ the second is the sharper. This gives the inequality (4.6) as we are assuming $\tau < 0$. □

Remark 4.2.4. The condition on the weights being generic implies that τ can not be zero. This is because $\tau = 0$ implies that $\sum 2\eta(x) - \alpha_1(x) - \alpha_2(x) = \Delta - 3b$, and if that happens then there is a U(2, 1)-parabolic Higgs subbundle, namely $(V \oplus 0, \Phi = 0)$, which is non-stable but semistable.

Remark 4.2.5. Note that the values m_1 and m_2 depend on ϖ_x and $\bar{d}_0$.

Theorem 4.2.6. *Fix d_0 and $\varpi = \{\varpi_x\}_{x \in D}$. Then the Poincaré polynomial of the critical submanifold $\mathcal{N}(d_0, \varpi)$ is*

$$P_t(\mathcal{N}(d_0, \varpi)) = (1+t)^{2g} \operatorname{Coeff}_{x^0 y^0} \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}} \right)$$

where m_1 and m_2 are the same as in Proposition 4.2.3.

Proof. Use MacDonald's formula for the Poincaré polynomial of the symmetric product (see [M]). □

Remark 4.2.7. The formula above is true also for $m_1 < 0$ or $m_2 < 0$. In this case $\mathcal{N}(d_0, \varpi)$ is empty and $P_t(\mathcal{N}(d_0, \varpi))$ is zero.

Now, in order to get the contribution of all the subvarieties of type $(1,1,1)$ to $P_t(\mathcal{U})$, we have to sum over all $d_0 \geq \bar{d}_0$ and all possibilities of ϖ. Substituing m_1 and m_2 from Proposition 4.2.3 and using the value of $\lambda_{\mathcal{N}(d_0,\varpi)}$ from Proposition 4.2.1, we get

$$
\begin{aligned}
P_t(\mathcal{U};(1,1,1)) &= \sum_{d_0,\varpi} t^{\lambda_{\mathcal{N}(d_0,\varpi)}} P_t(\mathcal{N}(d_0,\varpi)) \\
&= \sum_{d_0,\varpi} \left(t^{2g-2+2(b-\Delta)+4d_0+2(s-v)} \operatorname{Coeff}_{x^0y^0} \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}} \right)\right) \\
&= \operatorname{Coeff}_{x^0y^0} \left(\sum_{\varpi} \frac{t^{2g-2+2b-2\Delta+2s}(1+xt)^{2g}(1+yt)^{2g}}{(1-x)(1-xt^2)x^{b+2g-2}(1-y)(1-yt^2)y^{\Delta-2b+2g-2}} \cdot \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v_1}y^{v_2}} \cdot \frac{1}{1-t^4xy} \right) \\
&= \operatorname{Coeff}_{x^0y^0} \left(\frac{t^{2g-2+2b-2\Delta+2s}(1+xt)^{2g}(1+yt)^{2g}}{(1-x)(1-xt^2)x^{b+2g-2}(1-y)(1-yt^2)y^{\Delta-2b+2g-2}(1-t^4xy)} \cdot \sum_{\varpi} \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v_1}y^{v_2}} \right)
\end{aligned}
$$

Thus, we have to compute the following sum

$$
\sum_{\varpi_x} \frac{t^{4\bar{d}_0}x^{\bar{d}_0}y^{\bar{d}_0}}{t^{2v}x^{v_1}y^{v_2}}. \tag{4.9}
$$

The sum runs over all distributions $\varpi = \{\varpi_x\}_{x\in D}$. Note there are 2^s summands. The result depends on the system of weights α and η.

4.3 Computation of $P_t(\mathcal{U};(1,1,1))$ for one marked point.

From now on we consider the case where the surface X has one marked point to get more explicit formulas. This means that $D = x$, $x \in X$, $s = 1$. Denote $\alpha_i = \alpha_i(x)$ for $i = 1, 2$ and $\eta = \eta(x)$. We abbreviate ϖ_x to ϖ. There are two distributions of weights.

We have to consider the following cases for computing the sum (4.9),

Theorem 4.3.1. *The contribution to the Poincaré polynomial of the union of the submanifolds of type $(1,1,1)$ when $\Delta \equiv 0(3)$, $\tau < 0$ and D consist of one marked point depends on the possibilities for the weights of E shown in Table 4.1. They are the following,*

Table 4.1: Possibilities for the weights.

S_1	$\eta < \alpha_1 < \alpha_2$
$S_1(a)$	$\alpha_2 - \alpha_1 \geq \alpha_1 - \eta$
$S_1(b)$	$\alpha_2 - \alpha_1 < \alpha_1 - \eta$
S_3	$\alpha_1 < \alpha_2 < \eta$
$S_3(a)$	$\alpha_2 - \eta \geq \alpha_2 - \alpha_1$
$S_3(b)$	$\alpha_2 - \eta < \alpha_2 - \alpha_1$

S_2	$\alpha_1 < \eta < \alpha_2$
S_4	$\eta = \alpha_1 < \alpha_2$
S_5	$\alpha_1 < \alpha_2 = \eta$
S_6	$\alpha_1 = \alpha_2 = \eta$
S_7	$\eta < \alpha_1 = \alpha_2$
S_8	$\alpha_1 = \alpha_2 < \eta$

(i) For $S_1(a)$*:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2b-\frac{2\Delta}{3}+2g}\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{1+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}\,(1+t^2\,x\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)}\right).$$

(ii) For $S_1(b)$ *:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{-2+2b-\frac{2\Delta}{3}+2g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{1+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)}\right).$$

(iii) For S_2 *:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2b-\frac{2\Delta}{3}+2g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)}\right).$$

(iv) For $S_3(a)$ *:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2+2b-\frac{2\Delta}{3}+2g}\,(1+t^2)\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{3+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)(1-x\,y\,t^4)}\right).$$

(v) For $S_3(b)$*:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2b-\frac{2\Delta}{3}+2g}\,x^{1-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}\,(1+t^2\,x\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)}\right).$$

(vi) For S_4 *:*

$$\mathrm{Coeff}_{x^0y^0}\left(\frac{t^{2b-\frac{2\Delta}{3}+2g}\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,(1+t^2\,x)\,y^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-x\,y\,t^4)}\right).$$

(vii) For S_5 :

$$\operatorname{Coeff}_{x^0y^0}\left(\frac{t^{2b-\frac{2\Delta}{3}+2g}\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}\,(1+t^2\,y)}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-\,x\,y\,t^4)}\right).$$

(viii) For S_6 :

$$\operatorname{Coeff}_{x^0y^0}\left(\frac{2\,t^{2+2b-\frac{2\Delta}{3}+2g}\,x^{3-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{3+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-\,x\,y\,t^4)}\right).$$

(ix) For S_7 :

$$\operatorname{Coeff}_{x^0y^0}\left(\frac{2\,t^{-2+2b-\frac{2\Delta}{3}+2g}\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{1+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(-1+x)\,(-1+t^2\,x)\,(-1+y)\,(-1+t^2\,y)\,(1-\,x\,y\,t^4)}\right).$$

(x) For S_8 :

$$\operatorname{Coeff}_{x^0y^0}\left(\frac{2\,t^{-2+2b+4+4\frac{\Delta}{3}-2\Delta+2g}\,x^{2-b+\frac{\Delta}{3}-2g}\,(1+t\,x)^{2g}\,y^{3+2b-\frac{2\Delta}{3}-2g}\,(1+t\,y)^{2g}}{(1-x)\,(1-t^2\,x)\,(1-y)\,(1-t^2\,y)\,(1-\,x\,y\,t^4)}\right).$$

Proof. Compute the values of $\bar{d_0}$, v_1, v_2 and v for each one of the two possible distributions of the weights,ϖ, then we obtain the values for the sum in (4.9) for each case in Table 4.1.

The value for $\bar{d_0}$ depends on the distribution of the weights and is, in the case $\varpi = \mathrm{Id}$, $\bar{d_0} = \frac{\Delta}{3} + 1$ for all S_i except for $S_1(b)$ and S_7 where $\bar{d_0} = \frac{\Delta}{3}$. When $\varpi \neq \mathrm{Id}$, $\bar{d_0} = \frac{\Delta}{3}$ for all S_i except for $S_3(a)$, S_6 and S_8 where it is $\bar{d_0} = \frac{\Delta}{3} + 1$.

□

Remark 4.3.2. Polynomials above depend on $\Delta/3 - b$.

Remark 4.3.3. In cases S_4, S_5, S_6, S_7, S_8 one may get other polynomials for Δ not equivalent to $0(3)$.

4.4 Critical submanifolds of type $(1,2)$.

Following our discussion in Section 3.8 the critical submanifolds of type $(1,2)$ and $(2,1)$ can be identified with the moduli spaces $\mathcal{N}_{2g-2}(1,2,b+2g-2,a;\alpha_1,\alpha_2,\eta)$ and $\mathcal{N}_{2g-2}(2,1,a+4g-4,b;\alpha_1,\alpha_2,\eta)$respectively.

By Lemma 4.1.3, we may restrict to the case when $\tau < 0$. Then, the Morse function f has a minimum when $\gamma = 0$. Hence, for our analysis we only have to consider the critical subvarieties of type $(1,2)$, that is $(2g-2)$-stable parabolic triples in $\mathcal{N}_{2g-2}(1,2,b+2g-2,a;\alpha_1,\alpha_2,\eta)$.

Proposition 4.4.1. *The Morse index for a critical submanifold $\mathcal{N}$ of type $(2,1)$ or type $(1,2)$ is $\lambda_{\mathcal{N}} = 0$. In particular it does not depend on the weights. There is exactly one such critical submanifold, which is of type $(2,1)$ if $\tau > 0$ and of type $(1,2)$ if $\tau < 0$.*

Proof. This is clear since these subvarieties are minima for the Morse function. □

4.5 Computations of $P_t(\mathcal{U};(1,2))$ for one marked point and $\tau < 0$.

We particularise to the case where D consists of one point, $D = x$. By Lemma 4.1.3, we may assume $\Delta \equiv 0(3)$ and $\tau < 0$. Let ϖ be a fixed distribution of the weights over the marked point x. We define v_1, v_2 and v_3 in this section as follows:

$$v_1 = \begin{cases} 1 & \text{if} \quad \eta < \alpha_{\varpi(2)} \\ 0 & \text{otherwise} \end{cases}$$

$$v_2 = \begin{cases} 1 & \text{if} \quad \eta < \alpha_{\varpi(1)} \\ 0 & \text{otherwise} \end{cases}$$

$$v_3 = \begin{cases} 1 & \text{if} \quad \alpha_{\varpi(1)} < \alpha_{\varpi(2)} \\ 0 & \text{otherwise} \end{cases}$$

Let $\sigma > \sigma_m$ be a non-critical value for the moduli space $\mathcal{N}_\sigma(1,2,\bar{d}_1,\bar{d}_2;\eta,\alpha)$. For any ϖ, we define,

$$d_M = \left\lceil \frac{1}{3}\left(\Delta + \alpha_{\varpi(2)} + \eta - 2\alpha_{\varpi(1)} + \sigma\right) + 1 \right\rceil .$$

Proposition 4.5.1. *The Poincaré polynomial of the moduli space $\mathcal{N}_\sigma(1,2,\bar{d}_1,\bar{d}_2;\alpha,\eta)$ and one marked point is*

$$\operatorname{Coeff}_{x^0} \frac{(1+t)^{4g}(1+xt)^{2g}}{(1-t^2)(1-x)(1-xt^2)} \sum_{\varpi} x^{d_M - \bar{d}_1 + \bar{d}_2 - v_1} \left(\frac{t^{2\bar{d}_1 - 2\bar{d}_2 + 2v_2 + 2v_3 - 2d_M}}{1-t^{-2}x} - \frac{t^{-2\bar{d}_1 + 2g - 2v_3 + 4d_M}}{1-t^4 x} \right). \tag{4.10}$$

Proof. Rewrite theorem 6.5 from [GGM] for this concrete conditions. In [GGM] the Poincaré polynomial is computed under the assumption of generic distinct weights but here we apply the formula they give before assuming generic weights. □

Theorem 4.5.2. *The Poincaré polynomials of the critical submanifolds of type* $(1,2)$ *when* $\tau < 0$, $\Delta \equiv 0(3)$, *and* D *consist of one marked point depends on the possibilities for the weights of* E *given in Table 4.1. They are the following,*

(i) For $S_1(a)$ *and* S_7

$$\operatorname{Coeff}_{x^0}\Bigg(\frac{(1+t)^{4g}\,x^{1+2b-\frac{2}{3}\Delta-2g}(1+t\,x)^{2g}}{t^{4b+\frac{2}{3}\Delta-2g}(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}$$
$$\left(t^{6b}\,x-t^{4+6b}\,x+t^{2\Delta+2g}\,(1+x)-t^{4+2\Delta+2g}\,x\,(1+x)+t^{2+6b}\,(-1+x^2)\right)\Bigg).$$

(ii) For $S_1(b)$

$$\operatorname{Coeff}_{x^0}\Bigg(-\frac{(1+t)^{4g}\,(1+t^2)\,x^{1+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}}{t^{2+4b+\frac{2}{3}\Delta-2g}\,(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}$$
$$\left(t^{2+6b}-t^{2+2\Delta+2g}-t^{6b}\,x+t^{6+2\Delta+2g}\,x\right)\Bigg).$$

(iii) For S_2 *and* S_4

$$\operatorname{Coeff}_{x^0}\Bigg(-\frac{(1+t)^{4g}\,(1+t^2)\,x^{2+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}}{t^{2+4b+\frac{2}{3}\Delta-2g}\,(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}$$
$$\left(t^{4+6b}-t^{2\Delta+2g}-t^{2+6b}\,x+t^{4+2\Delta+2g}\,x\right)\Bigg).$$

(iv) For $S_3(a)$, S_6 *and* S_8

$$\operatorname{Coeff}_{x^0}\Bigg(-\frac{(1+t)^{4g}\,(1+t^2)\,x^{3+2b-\frac{2}{3}\Delta-2g}\,(1+t\,x)^{2g}}{t^{4+4b+\frac{2}{3}\Delta-2g}\,(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}$$
$$\left(t^{8+6b}-t^{2\Delta+2g}-t^{6+6b}\,x+t^{4+2\Delta+2g}\,x\right)\Bigg).$$

(v) For $S_3(b)$ *and* S_5

$$\operatorname{Coeff}_{x^0}\Bigg(\frac{(1+t)^{4g}\,x^{2+2b-\frac{2\Delta}{3}-2g}\,(1+t\,x)^{2g}}{t^{2+4b+\frac{2}{3}\Delta-2g}\,(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}$$
$$\left(t^{2+6b}\,x-t^{6+6b}\,x+t^{2\Delta+2g}\,(1+x)-t^{4+2\Delta+2g}\,x\,(1+x)+t^{4+6b}\,(-1+x^2)\right)\Bigg).$$

Proof. We only have to apply Proposition 4.5.1 using the different values for v_1 and v_2 on each case. Use also that $v_3 = 1$ if $\varpi = \mathrm{Id}$ and equal to zero otherwise. Hence we have different values for $\bar{d_M}$ depending on the distribution of the weights. These are, when $\varpi = \mathrm{Id}$, $\bar{d_M} = \frac{\Delta}{3} + 2g - 1$ fora all S_i except for $S_1(a)$ and S_7 where $\bar{d_M} = \frac{\Delta}{3} + 2g - 2$. And when $\varpi \neq \mathrm{Id}$, $\bar{d_M} = \frac{\Delta}{3} + 2g - 2$ for all S_i except for S_6 and S_8 where $\bar{d_M} = \frac{\Delta}{3} + 2g - 1$. □

Remark 4.5.3. Polynomials above depend on $\Delta/3 - b$. For example we rewrite (i)

$$\mathrm{Coeff}_{x^0}\left(\frac{(1+t)^{4g}\,x^{1+2b-\frac{2}{3}\Delta-2g}(1+t\,x)^{2g}}{t^{-2g}(-1+t^2)\,(t^2-x)\,(-1+x)\,(-1+t^2\,x)\,(-1+t^4\,x)}\right.$$

$$\left.\left(t^{2b-\frac{2}{3}\Delta}\,x - t^{4+2b-\frac{2}{3}\Delta}\,x + t^{\frac{4}{3}\Delta-4b+2g}\,(1+x) - t^{4+\frac{4}{3}\Delta-4b+2g}\,x\,(1+x) + t^{2+2b-\frac{2}{3}\Delta}\,(-1+x^2)\right)\right).$$

4.6 Poincaré polynomial of $\mathcal{U}$ for the case of one marked point

Summarising, we are using Morse-Bott theory in order to compute the Poincaré polynomial of the moduli space of stable parabolic $U(2,1)$-Higgs bundles with degrees $a = \deg(V)$ and $b = \deg(W)$. Therefore we have described the critical subvarieties of $\mathcal{U}$ for the Morse function f. These consist of several submanifolds of type $(1,1,1)$ parametrised by (d_0, ϖ) and, depending on τ, and one submanifold corresponding to the minima of f, which is of type $(1,2)$ when $\tau < 0$, and of type $(2,1)$ when $\tau > 0$.

Theorem 4.6.1. *Let X be a compact Riemann surface with one marked point x and $D = x$ the corresponding effective divisor. Let $\mathcal{U}(2,1,a,b;\alpha,\eta)$ be the moduli space of parabolic* U(2, 1)*-Higgs bundles on X with fixed degrees a, b, and generic weights α, η. Let τ be the parabolic Toledo invariant for the bundles in $\mathcal{U}(2,1,a,b;\alpha,\eta)$. Then, the Poincaré polynomial of $\mathcal{U}(2,1,a,b;\alpha,\eta)$ is given by*

$$P_t(\mathcal{U}(2,1,a,b;\alpha,\eta)) = \begin{cases} P_t(\mathcal{U};(1,1,1)) + P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b)) & \text{if } \tau < 0 \\ P_t(\mathcal{U};(1,1,1)) + P_t(\mathcal{N}_{2g-2}(1,2,b+2g-2,a)) & \text{if } \tau > 0 \end{cases} \tag{4.11}$$

Let us see what Poincaré polynomial of the moduli space of parabolic U(2, 1)-Higgs bundles we obtain when D consists of only one marked point by using Sections 4.3 and 4.5, for specific values of a, b, weights and genus g.

Note that in order to give an example we fix a, b such that $\Delta = a + b \equiv 0(3)$ and $\tau < 0$.

Fix $g = 1$, degrees $a = b = 0$, and $\alpha_1 < \alpha_2 < \eta$ such that $\eta - \alpha_2 < \alpha_2 - \alpha_1$, that is, we are in case $S_3(b)$ of Table 4.1.

The contribution from the critical subvarieties of type $(1,1,1)$ is

$$P_t(\mathcal{U};(1,1,1)) = t^2 + 2t^3 + t^4.$$

The contribution for the critical subvariety of type $(1,2)$ is

$$P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b)) = 1 + 4t + 6t^2 + 4t^3 + t^4.$$

Note that by the Küneth Theorem, the polynomial is symmetric. Incidentally, this coincides with the Poincaré polynomial of a 4-torus. It would be interesting to see wether it is the same variety.

Hence, the Poincaré polynomial for $\mathcal{U}$ when $a = b = 0$, $g = 1$, and $\alpha_1 < \alpha_2 < \eta$ such that $\eta - \alpha_2 < \alpha_2 - \alpha_1$, is

$$P_t(\mathcal{U}) = 1 + 4t + 7t^2 + 6t^3 + 2t^4.$$

Observe that this does not satisfy Poincaré duality, since $\mathcal{U}$ is non-compact.

As an example of the phenomena we have talked about above, i.e. that we get different polynomials for different weights, we give another example for same genus and degrees but for case S_5 in Table 4.1.

The contribution from the critical subvarieties of type $(1,1,1)$ is

$$P_t(\mathcal{U};(1,1,1)) = t^2,$$

and from the critical subvariety of type $(1,2)$ is again

$$P_t(\mathcal{N}_{2g-2}(2,1,a+4g-4,b)) = 1 + 4t + 6t^2 + 4t^3 + t^4.$$

Hence, the Poincaré polynomial for $\mathcal{U}$, when $a = b = 0$, $g = 1$, one marked point and, $\alpha_1 < \alpha_2 = \eta$, is

$$P_t(\mathcal{U}) = 1 + 4t + 7t^2 + 4t^3 + t^4$$

Proposition 4.6.2. *The complex dimension of the moduli space of parabolic $U(2,1)$-Higgs bundles is $1 + 9(g-1) + \sum_{x\in D}(3-c)$ where c is the number of weights $\alpha_i(x)$ equal to $\eta(x)$.*

Proof. Rewrite Proposition 3.3.1 from Section 3.3. □

Hence, in the examples above the real dimension of $\mathcal{U}$ is 6 and does not coincides with the degree of the polynomials. Also it is interesting to note the fact that one of the polynomials satisfies Poincaré duality and the other does not. No surprise since $\mathcal{U}$ is smooth but non-compact.

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Introducción

Sea X una superficie de Riemann compacta de género $g \geq 0$ y sean $\{x_1, \ldots, x_s\}$ un conjunto de puntos marcados sobre la superficie con $D = x_1 + \cdots + x_s$ su correspondiente divisor efectivo. Un fibrado parabolico se define como un fibrado vectorial junto con una filtración con pesos en cada fibra sobre $x \in D$, i.e.

$$\begin{aligned} E_x &= E_{x,1} \supset \cdots \supset E_{x,r(x)} \\ 0 &\leq \alpha_1(x) < \cdots < \alpha_{r(x)}(x) < 1, \quad x \in D, \end{aligned}$$

Los fibrados parabólicos fueron introducidos por Seshadri en [Se] con el objeto de obtener una desingularización del espacio de moduli de fibrados vectoriales semiestables de rango dos y grado cero [Se2]. El *grado parabólico* y la *pendiente parabólica* se definen de la siguiente forma.

$$\begin{aligned} \operatorname{pardeg}(E) &= \deg(E) + \sum_{x \in D} \sum_{i=1}^{r(x)} k_{x,i}\alpha_i(x), \\ \operatorname{par}\mu(E) &= \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)}, \end{aligned}$$

donde $\deg(E)$ es el grado de E, $\operatorname{rk}(E)$ el rango de E y $k_{x,i} = \dim E_{x,i}/E_{x,i+1}$ es la *multiplicidad* del peso α_i en el punto marcado x. El fibrado parabólico tiene las *filtraciónes llenas* si todas las multiplicidades $k_{x,i} = 1$ para todo peso α_i sobre cualquier punto marcado x. El fibrado parabólico se dice *(semi)estable* si $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (respectivamente$\operatorname{par}\mu(E') \leq \operatorname{par}\mu(E)$) para todo subfibrado propio de E' de E. El fibrado parabólico E se dice *poliestable* si E se descompone en suma directa de fibrados parabólicos estables con la misma pendiente parabólica. Con esta noción de estabilidad Mehta y Seshadri construyeron el espacio de moduli de fibrados parabólicos estables, con estructuras parabólicas y grados fijos, usando la teoría geométrica de invariantes de Mumford [MF]. Este espacio de moduli resulta ser una variedad proyectiva normal, que es además lisa en el caso de *pesos genéricos*, esto es, para aquellos pesos que no permitan la existencia de fibrados estrictamente semistables en el espacio de moduli. Resulta además que, para pesos genéricos suficientemente cercanos, los espacios de moduli son isomorfos.

Mehta y Seshadri en [MS] mostraron que, el espacio de moduli de fibrados parabólicos poliestables con grado parabólico cero y con estructura parabolica fija sobre X con género $g \geq 2$, se identifica con el espacio de moduli de representaciones unitarias del grupo fundamental de $X - D$ con la holonomía, alrededor de los puntos marcados, determinada por la estructura parabólica. Otra demostración de este teorema, usando teorías gauge, ha sido dada por Biquard [B] y otros [P, Bo, NaSt]. El teorema de Mehta y Seshadri generaliza un teorema de Narasimhan y Seshadri [NS] que identifica, el espacio de moduli de fibrados vectoriales poliestables de grado cero sobre una superficie de Riemann compacta, con el espacio de representaciones unitarias del grupo fundamental de la superficie de Riemann.

Sea K el fibrado canónico de X. Un *fibrado de Higgs* $\mathrm{GL}(n\mathbb{C})$ *parabólico* es un par (E, Φ) donde E es un fibrado parabólico, y $\Phi : E \longrightarrow E \otimes K(D)$ es un *homomorfismo parabólico estricto* i.e. Φ es un endomorphismo meromorfo con valores en las 1-formas y polos simples alrededor de los puntos marcados cuyo residuo en $x \in D$ es nilpotente con respecto a la filtración. Un fibrado de Higgs parabólico $\mathrm{GL}(n, \mathbb{C})$ es *estable* si $\mathrm{par}\mu\,(E') < \mathrm{par}\mu\,(E)$ para todo $E' \subset E$ subfibrado parabólico propio y Φ-invariante, i.e.$\Phi(E') \subset E' \otimes K(D)$. Se dice que es *semiestable* si en la desigualdad para las pendientes parabólicas permitimos también la igualdad y, se dice *poliestable* si se descompone en suma directa de fibrados de Higgs $\mathrm{GL}(n, \mathbb{C})$ parabólicos con la misma pendiente parabólica.

El espacio de moduli de fibrados de Higgs $\mathrm{GL}(n, \mathbb{C})$ parabólicos, $\mathcal{M}$, fue construido usando teoría geométrica de invariantes por Yokogawa en [Y1, Y2], quien también muestra que, para pesos genéricos, este espacio de moduli es una variedad compleja cuasiproyectiva lisa e irreducible. Análogamente al caso no parabólico, el espacio de moduli $\mathcal{M}$ contiene al fibrado cotangente del espacio de moduli de fibrados parabólicos estables.

Simpson en [S2] prueba que el espacio de moduli de fibrados de Higgs $\mathrm{GL}(n, \mathbb{C})$ parabólicos poliestables de grado cero puede identificarse con el espacio de moduli de representaciones en $\mathrm{GL}(n, \mathbb{C})$ del grupo fundamental de la superficie con puntos marcados, con holonomia fija y compacta alrededor de los puntos marcados. Esto extiende a superficies de Riemann no compactas la teoría de fibrados de Higgs y representaciones del grupo fundamental desarrollada por Hitchin, Simpson, Donaldson y Corlette [H, S1, D2, C].

Esta tesis se dedica al estudio de fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos . Un *fibrado de Higgs* $\mathrm{U}(p,q)$ *parabólico* es un fibrado de Higgs $\mathrm{GL}(n,\mathbb{C})$ parabólico de la forma,

$$E = V \oplus W \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \to (V \oplus W) \otimes K(D),$$

donde $\beta : W \to V \otimes K(D)$ y $\gamma : V \to W \otimes K(D)$ son homomorfismos parabólicos estrictos.

Un fibrado de Higgs $\mathrm{U}(p,q)$ parabólico se dice *estable* si la condicion para la pendiente parabólica se satisface para todo subfibrado propio Φ-invariante $E' \subset E$ tal que $E' = V' \oplus W'$ con $V' \subset V$ y $W' \subset W$ i.e para todos los subfibrados parabólicos $V' \subset V$ y $W' \subset W$ tales que,

$$\begin{aligned} \gamma(V') &\subset W' \otimes K(D) \\ \beta(W') &\subset V' \otimes K(D). \end{aligned}$$

La *semiestabilidad* se define permitiendo la igualdad en las pendientes parabólicas. El fibrado se dice poliestable si se puede escribir como suma directa de fibrados parabólicos estables con la misma pendiente.

Sea $\mathcal{U}(p,q,a,b;\alpha,\eta)$ el *espacio de moduli de fibrados de Higgs* $\mathrm{U}(p,q)$ *parabólicos estables* con rangos $\mathrm{rk}\,(V) = p$, $\mathrm{rk}\,(W) = q$, grados $\deg(V) = a$, $\deg(W) = b$ y pesos α y η para V y W. El espacio de moduli $\mathcal{U}(p,q,a,b;\alpha,\eta)$ es una subvariedad cerrada del espacio de moduli $\mathcal{M}(p+q, a+b; \alpha \cup \eta)$, donde $\alpha \cup \eta$ son los pesos correspondientes a la suma directa de los fibrados parabolicos V y W, y es isomorfo al espacio de moduli de representaciones irreducibles en $\mathrm{U}(p,q)$ del grupo fundamental de la superficie con las clases de holonomía, alrededor de los puntos marcados, fijadas en $\mathrm{U}(p \times \mathrm{U}(q)$.

El objeto principal de esta tesis es el cálculo del número de componentes conexas de $\mathcal{U}(n,1,a,b;\alpha,\eta)$. El caso no parábolico fue resuelto en [BGG] para fibrados de Higgs $\mathrm{U}(p,q)$.

El *invariante de Toledo parabólico* τ para fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos se define como,

$$\tau = 2\frac{q\,\mathrm{pardeg}\,(V) - p\,\mathrm{pardeg}\,(W)}{p+q}$$

usando estabilidad se obtiene que

$$|\tau| < \min\{p,q\}(2g-2+s),$$

generalizando la desigualdad de Milnor–Wod para el caso no parabólico.

Cuando restringimos $q = 1$ encontramos una cota más precisa para el invariante de Toledo, de hecho nosotros encontramos el valor máximo τ_L de τ, que cumple $\tau_L \leq (2g-2+s)$, alcanzando la igualdad cuando no hay puntos marcados.

El principal resultado de esta tesis es el siguiente.

Teorema. *Sea X una superficie de Riemann compacta con género $g \geq 1$, y sean α y η pesos genéricos. Entonces, el espacio de moduli $\mathcal{U}(n,1,a,b;\alpha,\eta)$ de fibrados de Higgs $\mathrm{U}(n,1)$ parabólicos con filtraciones llenas es no vacío y conexo si y sólo si $|\tau| < \tau_L$.*

Nuestra aproximación al estudio de $\mathcal{U}$ combina las técnicas usadas en [BGG] para el estudio de fibrados de Higgs $\mathrm{U}(p,1)$ así como el usado en [GGM] para estudiar la topología del espacio de moduli de fibrados de Higgs $\mathrm{GL}(3,\mathbb{C})$ parabólicos. La principal herramienta para el estudio de estos espacios de moduli es el uso de la teoría de Morse introducido por Hihtchin en [H]: la norma L^2 del campo de Higgs es la aplicación momento f asociada a una acción Hamiltoniana de S^1 en el moduli de soluciones a las ecuaciones de Hitchin. Tal función f es propia y acotada inferiormente, por lo tanto si el subespacio de mínimos locales de f es conexo también lo es $\mathcal{U}$.

El espacio de minimos locales de f está relacionado con otro tipo de obejtos parabólicos: las ternas parabólicas. Una terna parabólica es una terna $T = (E_1, E_2, \phi)$ formada por dos fibrados parabólicos E_1 y E_2 y un morfismo estrictamente parabólico $\phi : E_2 \to E_1(D)$. Para un parámetro $\sigma \in \mathbb{R}$ definimos la σ-pendiente parabólica como

$$\operatorname{pardeg}(T) = \frac{\operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)} + \sigma \frac{\operatorname{rk}(E_2)}{\operatorname{rk}(E_1) + \operatorname{rk}(E_2)}.$$

Una terna parabólica se dice *σ-stable* si la condición $\operatorname{par}\mu_\sigma(T') < \operatorname{par}\mu_\sigma$ se satisface para toda subterna T' de T. La semiestabilidad se define permitiendo la igualdad en la condición sobre las σ-pendientes y, la terna se dice poliestable si se descompone en suma directa de ternas σ-estables con la misma σ-pendiente.

Denotamos por $\mathcal{N}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ al espacio de moduli de ternas σ-estables con rangos $\operatorname{rk}(E_1) = r_1$, $\operatorname{rk}(E_2) = r_2$, grados $\deg(E_1) = d_1$, $\deg(E_2) = d_2$, y pesos α^1 y α^2.

La subvariedad de mínimos locales de f se identifica con un cierto espacio de moduli de ternas parabólicas para $\sigma = 2g - 2$. Centramos, pues, nuestro estudio en el cálculo del número de componentes conexas de este espacio de moduli. Este cálculo necesita conocer a su vez como cambia el moduli de ternas estables cambia cuando cambiamos el valor del parametro σ y mantenemos fijos rangos, grados y pesos. Fijamos, pues, el tipo topológico de la terna, es decir fijamos $(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$. Llamamos σ_c valor crítico del parámetro σ a un valor tal que existe una terna σ-semistable para el tipo topólogico fijado. Llamamos *flip loci* al subconjunto $\mathcal{N}_{\sigma_c^\pm}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ de ternas parabólicas $\sigma_c^\pm$-estables, donde por $\sigma_c^\pm$ entendemos $\sigma_c^+ = \sigma_c + \epsilon$, $\sigma_c^- = \sigma_c - \epsilon$ con $\epsilon \in \mathbb{R}$, $\epsilon > 0$ y suficientemente pequeño como para que no exista otro valor crítico en $[\sigma_c, \sigma_c^\pm]$. Este es el espacio que ha de ser añadido al espacio de moduli $\mathcal{N}_{\sigma_c}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ cuando cruzamos un valor crítico de σ. Nosotros probamos que para $\sigma \geq 2g - 2$ la codimension del flip loci es positiva, por lo tanto, los espacios de moduli $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ son biracionales para todos los valores de $\sigma \geq 2g - 2$.

Teorema. *Sea X una superficie de Riemann compacta con un conjunto finito de puntos marcados. Sea $g \geq 1$ el género de X. Sea $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ el espacio de moduli de ternas parabólicas σ-estables con filtraciones llenas. Entonces, para $\sigma \in [2g - 2, \sigma_L)$, los espacios de moduli $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ son biracionales.*

De modo similar a como se hizo para el invariante de Toledo parabólico, usando la estabilidad uno puede mostrar que en el caso $r_1 \neq r_2$ existe una cota superior para el parámetro de estabilidad

$$\sigma < \sigma_M.$$

Esta cota puede también ser mejorada. En la Proposición 2.7.9 calculamos su valor máximo y lo denotamos por σ_L. En el caso $r_1 = r_2$, σ no está acotado superiormente, en ese caso σ_L denota el mayor valor critico para σ.

Este valor máximo σ_L es tal que para σ_L^+ el espacio de moduli $\mathcal{N}_{\sigma_L^+}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$ es vacío y para σ_L^- no vacío.

La biracionalidad entre estos espacios de moduli implica que es suficiente contar el número de componentes para el espacio de moduli $\mathcal{N}_{\sigma_L^-}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$. Damos la descripcion explicita de este espacio de moduli para $r_1 = n$ y $r_2 = 1$, que nos da el siguiente teorema.

Teorema. *Sea X una superficie de Riemann compacta con un conjunto finito de puntos marcados y género $g \geq 1$. El espacio de moduli $\mathcal{N}_{\sigma_L^-}(n, 1, d_1, d_2; \alpha^1, \alpha^2)$ de ternas parabólicas σ-estables con pesos genéricos y filtraciones llenas es una variedad algebraica irreducible*

La aplicación momento f también define una función de Bott–Morse perfecta sobre el espacio de moduli. El cálculo de la serie de Poincaré y de los índices de todas las subvariedades críticas de f nos dan los números de Betti de $\mathcal{U}$. Las subvariedades críticas de f

son: los minimos, identificados como ternas parabólicas de rangos $(2,1)$ y $(1,2)$, y *cadenas parabólicas* con rangos $(1,1,1)$, donde una cadena parabólica esta definida como una n-tupla de fibrados parabólicos E_i junto con una $(n-1)$-tupla de homomorfismos estrictamente parabólicos $\phi_i : E_i \longrightarrow E_{i-1}(D)$. Por lo tanto necesitamos calcular el polinomio de Poincaré del espacio de moduli de cadenas parabólicas. En el caso de rangos $p = 2$ y $q = 1$, este cálculo puede realizarse gracias a que el espacio de moduli de cadenas parabólicas es isomorfo a $\mathrm{Jac}\,(X)^d \times S^{m_1}X \times S^{m_2}X$ para ciertos d, m_1 y m_2, y cuyo polinomio de Poincaré es conocido [M]. Para (p,q) arbitrarios este cálculo es mucho más difícil y está fuera del alcance de esta tesis. El polinomio de Poincaré del espacio de moduli de ternas de rangos $(2,1)$ y $(1,2)$ son calculados usando los resultados de [GGM].

Por lo tanto, calculamos el polinomio de Poincaré de $\mathcal{U}(2,1,a,b;\alpha,\eta)$ para el caso de un punto marcado, ver Teorema 4.6.1. Como mencionamos antes el espacio de moduli $\mathcal{U}(2,1,a,b;\alpha,\eta)$ es una subvariedad del espacio de moduli $\mathcal{M}(3,a+b,\alpha\cup\eta)$. Sabemos que, el espacio de moduli $\mathcal{M}(3,a+b,\alpha\cup\eta)$ de fibrados de Higgs $\mathrm{GL}(3,\mathbb{C})$ parabólicos tiene los mismos números de Betti para diferentes grados y pesos (ver [GGM, T]). Nuestros cálculos dan un contraejemplo sobre este tipo de fenómeno para el caso $\mathrm{U}(2,1)$ pues, como vemos en el Teorema 4.6.1, los números de Betti del espacio de móduli $\mathcal{U}(2,1,a,b;\alpha,\eta)$ dependen de la estructura parabólica.

Damos ahora una description general de la tesis. En el capítulo 1 estudiamos algunos hechos básicos sobre fibrados parabólicos y homomorfismos parabólicos. Introducimos la correspondencia dada por el teorema de Mehta y Seshadri.

El capítulo 2 está dedicado a las ternas parabólicas y a sus homomorfismos parabólicos. Recordamos la teoría de extensiones y deformaciones de ternas parabólicas estudiada en [GGM] así como la noción de *flip loci* para el estudio de cómo varia el espacio de moduli de ternas parabólicas al variar el parametro de estabilidad σ. Damos una cota para la codimensión del flip loci. Para ello hacemos uso de la correspondencia entre ternas parabólicas estables y soluciones a las ecuaciones vorticiales parabólicas. Probamos que la codimensión del flip loci es positiva para $\sigma \in [2g-2, \sigma_L)$ y cero para $\sigma = \sigma_L$. Para $\sigma \geq \sigma_L$ espacio de moduli es vacío y es no vacío para σ_L^-. Finalmente describimos explícitamente el espacio de moduli $\mathcal{N}_{\sigma_L^-}(n,1,d_1,d_2;\alpha^1,\alpha^2)$.

El capítulo 3 se dedica a los fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos. Estudiamos aquí las cotas del invariante de Toledo parabólicos y damos su valor máximo cuando $q=1$. También mostramos las técnicas de la teoría de Morse necesarias para el estudio de $\mathcal{U}$, y mostramos la correspondencia entre espacios de moduli $\mathcal{N}_\sigma(r_1,r_2,d_1,d_2;\alpha^1,\alpha^2)$ y mínimos de la función de

Morse f. Este hecho conecta con el estudio sobre el moduli de ternas prabólicas del capítulo anterior. Mostramos también la relación entre $\mathcal{U}(p,q,a,b;\alpha,\eta)$ y el espacio de moduli de representaciones en $\mathrm{U}(p,q)$ del grupo fundamental de la superficie de Riemann no compacta $X-D$ con holonomia en $\mathrm{U}(p)\times\mathrm{U}(q)$ fija alrededor de los puntos de D.

Finalmente en el capítulo 4 fijamos los rangos $p=2$ y $q=1$, y usamos teoría de Bott–Morse para calcular los números de Betti de $\mathcal{U}$ cuando $p+q=3$, y un punto marcado.

Conclusiones

En esta tesis se estudian fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos sobre una superficie de Riemann compacta X con un conjunto finito de puntos marcados D. Estos objetos están en correspondencia con las representaciones del grupo fundamental de la superficie menos los puntos marcados en $\mathrm{U}(p,q)$, con las clases de holonomia alrededor de cada punto marcado fijas y compactas.

El objetivo es calcular el número de componentes conexas del móduli de fibrados de Higgs $\mathrm{U}(n,1)$ parabólicos y en el caso en que $n=2$ dar además su polinomio de Poincaré. La estrategia a seguir para ello consiste en usar técnicas de la teoría de Bott–Morse ya introducidas en este tipo de problemas por Hitchin en [H]. Con estas técnicas estudiar la conexión de este espacio de moduli se reduce a estudiar ciertos espacios de moduli de ternas parabólicas, sobre los cuales damos un resultado sobre la biracionalidad entre los espacios de moduli según la variacion del parámetro de estabilidad y, otro sobre la irreducibilidad en el caso de que fijemos el rango de uno de los fibrados que componen la terna igual a uno. Las ternas parabólicas fueron introducidas por Biquard y García-Prada en [BG] en su estudio de las ecuaciones vorticiales parabólicas y los instantones de energía infinita. Mucha parte de nuestro trabajo consiste en dar las propiedades de conexión de los espacios de moduli de ternas parabólicas.

Nuestra aproximación al estudio de fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos combina las técnicas usadas por Bradlow, García-Prada y Gothen en [BGG] para el caso no parabólico asícomo aquellas usadas por García-Prada, Gothen y Muñoz en [GGM] para estudiar la topología de los fibrados de Higgs $\mathrm{GL}(3,\mathbb{C})$ parabólicos.

Nuestros principales resultados incluyen el contar el número de componentes conexas del espacio de moduli de fibrados de Higgs $\mathrm{U}(n,1)$ parabólicos. Gracias a la correspondencia, antes mencionada, entre espacios de moduli $\mathcal{U}$ de fibrados de Higgs $\mathrm{U}(p,q)$ parabólicos y el moduli de representaciones en $\mathrm{U}(p,q)$ del grupo fundamental de la superficie menos los puntos marcados con holonomia en $\mathrm{U}(p) \times \mathrm{U}(q)$ fija alrededor de los puntos marcados, obtenemos también el número de componentes conexas del moduli de representaciones en $\mathrm{U}(n,1)$ del grupo fundamental de $X - D$.

Para el caso de tener un sólo punto marcado en X, i.e. $D = x$, calculamos el polinomio de Poincaré del espacio de moduli de fibrados de Higgs $U(2,1)$ parabólicos. Obtenemos que, a diferencia de lo que ocurre para el moduli de fibrados de Higgs $\mathrm{GL}(3,\mathbb{C})$ parabólicos, del cual $\mathcal{U}$ es una subvariedad cerrada, el polinomio de Poincaré depende de los pesos que fijemos.

Reunido el tribunal que suscribe en el día de la fecha, acordó calificar la presente Tesis doctoral con Sobresaliente cum Laude.

MADRID 19 de Mayo de 2006

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